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Clauſtra patent *cæli*, rerumq; immobilis ordo.
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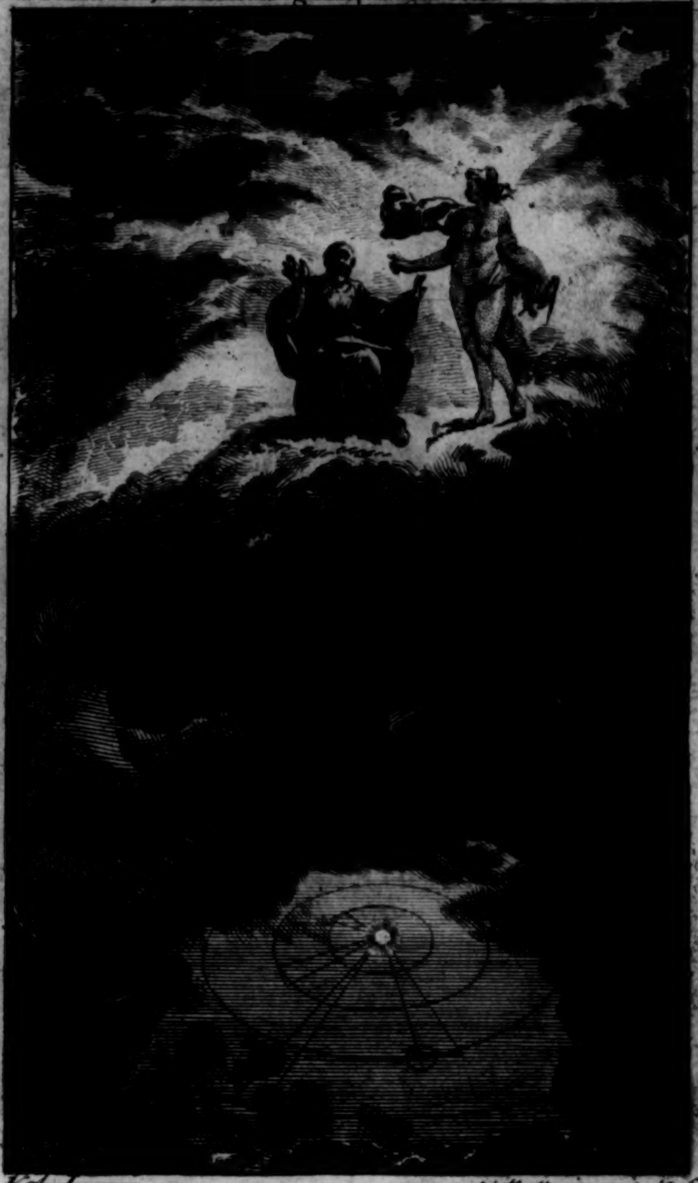
Scandere, ſublimis genij conceſſit acumen. *Dr. Halley.*



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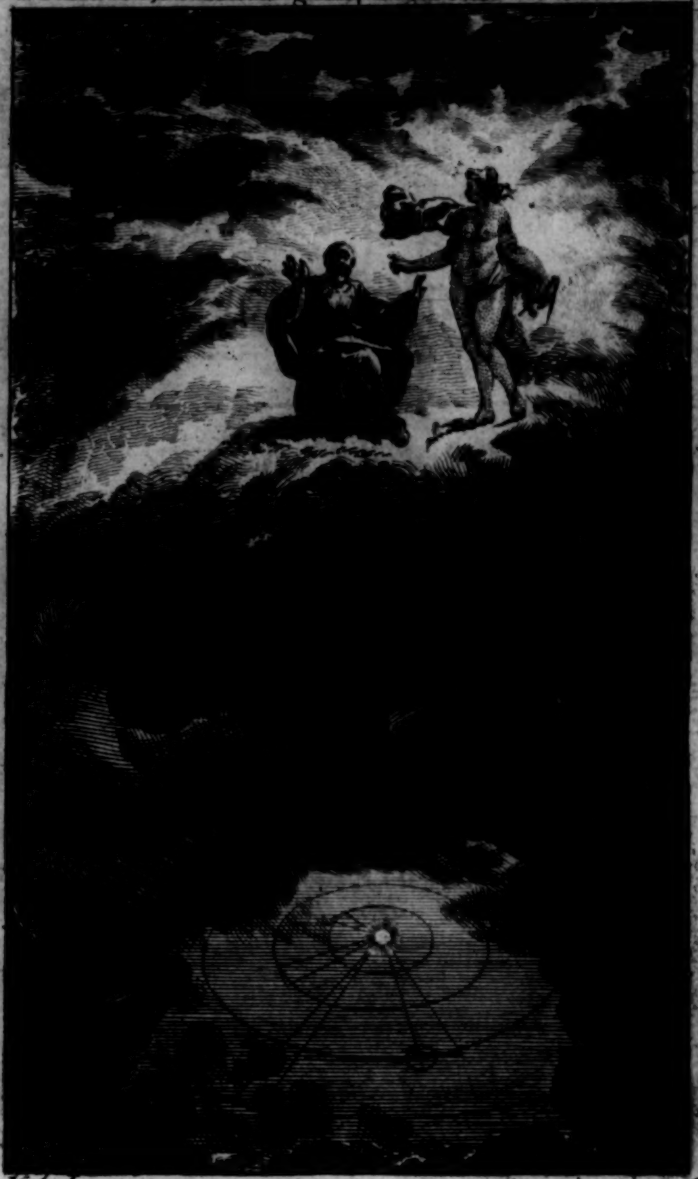

Vol. I. *A. Motte invenit & fecit.*

9Rh

THE MATHEMATICAL PRINCIPLES OF Natural Philoſophy.

By Sir *ISAAC NEWTON*.

Tranſlated into *Engliſh* by ANDREW MOTTE.

To which are added,

The Laws of the MOON's Motion, according to Gravity.

By JOHN MACHIN *Aſtron. Prof. Greſh.* and *Secr. R. Soc.*

IN TWO VOLUMES.

LONDON:
Printed for BENJAMIN MOTTE, at the *Middle-Temple-Gate*, in *Fleetſtreet*.
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MVSEVM
BRITAN
NICVM

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TO
Sir HANS SLOANE,
BARONET,
PRESIDENT
OF THE
College of Physicians,
AND OF THE
ROYAL SOCIETY.

SIR,

THE generous zeal You always ſhew for whatever tends to the progreſs and advancement of Learning, both demands and receives the univerſal acknowledgments of all who profeſs or value its ſeveral branches.

THEY juſtly admire that amidſt a cloſe attendance on the cares of your Profeſſion, in which You now fill the moſt honourable Seat, You are indefatigably promoting the improvement of natural knowledge, by carrying on ſome laudable deſigns of your own, by aſſiſting and encouraging others, and by adding new ſtores to that immenſe treaſure, already brought into your extenſive collection, of whatever is rare and valuable in nature or art.

YOUR beneficent diſpoſition to countenance and favour Science and Literature, has procured You the eſteem of the Learned over all the World; and has induced a Body of Men, the moſt eminent for their skill and diligence in all uſeful enquiries, and in purſuing diſcoveries for the public good, to make choice of You, to ſupply the place of Him, whoſe Name will be an everlaſting honour to our age and nation.

To whom therefore but to You ſhould I offer to inſcribe the tranſlation of the moſt celebrated Work of your Illuſtrious Predeceſſor? which, on account of its incomparable Author, and from the dignity of the Subject, claims and deſerves your acceptance, even tho' it paſs'd thro' my hands: a leſs valuable Piece I ſhould not have preſumed to preſent You with. I am, with the greateſt reſpect,

SIR,

Your moſt obedient, and

moſt humble Servant,

Andr. Motte.

THE

Author's Preface.

SINCE the ancients (as we are told by Pappus) *made great account of the ſcience of Mechanics in the inveſtigation of natural things; and the moderns, laying aſide ſubſtantial forms and occult qualities, have endeavoured to ſubject the phænomena of nature to the laws of mathematics; I have in this treatiſe cultivated Mathematics, ſo far as it regards Philoſophy. The ancients conſidered Mechanics in a twofold reſpect; as rational, which proceeds accurately by demonſtration, and practical. To practical Mechanics all the manual arts belong, from which Mechanics took its name. But as artificers do not work with perfect accuracy, it comes to paſs that Mechanics is ſo diſtinguiſhed from Geometry, that what is perfectly* *accurate is called Geometrical, what is leſs ſo is called Mechanical. But the errors are not in the art, but in the artificers. He that works with leſs accuracy, is an imperfect Mechanic, and if any could work with perfect accuracy, he would be the moſt perfect Mechanic of all. For the deſcription of right lines and circles, upon which Geometry is founded, belongs to Mechanics. Geometry does not teach us to draw theſe lines, but requires them to be drawn. For it requires that the learner ſhould firſt be taught to deſcribe theſe accurately, before he enters upon Geometry; then it ſhews how by theſe operations problems may be ſolved. To deſcribe right lines and circles are problems, but not geometrical problems. The ſolution of theſe problems is required from Mechanics; and by Geometry the uſe of them, when ſo ſolved, is ſhewn. And it is the glory of Geometry that from thoſe few principles, fetched from without, it is able to produce ſo many things. Therefore Geometry is founded in mechanical practice, and is nothing but that part of univerſal Mechanics which accurately propoſes and demonſtrates the art of meaſuring. But ſince the manual arts are chiefly*

chiefly converſant in the moving of bodies, it comes to paſs that Geometry is commonly referred to their magnitudes, and Mechanics to their motion. In this ſenſe Rational Mechanics will be the ſcience of motions reſulting from any forces whatſoever and of the forces required to produce any motions, accurately propoſed and demonſtrated. This part of Mechanics was cultivated by the ancients in the Five Powers which relate to manual arts, who conſidered gravity (it not being a manual power) no otherwiſe than as it moved weights by thoſe powers. Our deſign not reſpecting arts but philoſophy, and our ſubject, not manual but natural powers, we conſider chiefly thoſe things which relate to gravity, levity, elaſtic force, the reſiſtance of fluids, and the like forces whether attractive or impulſive. And therefore we offer this work as mathematical principles of philoſophy. For all the difficulty of philoſophy ſeems to conſiſt in this, from the phænomena of motions to inveſtigate the forces of Nature, and then from theſe forces to demonſtrate the other phænomena. And to this end, the general propoſitions in the firſt and ſecond book are directed. In the third book we give an example of this in the explication of *the Syſtem of the World. For by the propoſitions mathematically demonſtrated in the firſt books, we there derive from the celeſtial phænomena, the forces of Gravity with which bodies tend to the Sun and the ſeveral Planets. Then from theſe forces by other propoſitions, which are alſo mathematical, we deduce the motions of the Planets, the Comets, the Moon, and the Sea. I wiſh we could derive the reſt of the phænomena of Nature by the ſame kind of reaſoning from mechanical principles. For I am induced by many reaſons to ſuſpect that they may all depend upon certain forces by which the particles of bodies, by ſome cauſes hitherto unknown, are either mutually impelled towards each other and cohere in regular figures, or are repelled and recede from each other; which forces being unknown, Philoſophers have hitherto attempted the ſearch of Nature in vain. But I hope the principles here laid down will afford ſome light either to that, or ſome truer, method of Philoſophy.*

In the publication of this Work, the moſt acute and univerſally learned Mr. Edmund Halley *not only aſſiſted me with his pains in correcting the preſs and taking care of the ſchemes, but it was to his*

his ſolicitations that its becoming publick is owing. For when he had obtained of me my demonſtrations of the figure of the celeſtial orbits, he continually preſſed me to communicate the ſame to the Royal Society; *who afterwards by their kind encouragement and entreaties, engaged me to think of publiſhing them. But after I had begun to conſider the inequalities of the lunar motions, and had entered upon ſome other things relating to the laws and meaſures of gravity, and other forces; and the figures that would be deſcribed by bodies attracted according to given laws; and the motion of ſeveral bodies moving among themſelves; the motion of bodies in reſiſting mediums; the forces, denſities, and motions of mediums; the orbits of the Comets, and ſuch like; I put off that publication till I had made a ſearch into thoſe matters, and could put out the whole together. What relates to the Lunar motions (being imperfect) I have put all together in the corollaries of prop.* 66. *to avoid being obliged to propoſe and diſtinctly demonſtrate the ſeveral things there contained in a method more prolix than the ſubject deſerved, and interrupt the ſeries of the ſeveral propoſitions. Some things* *found out after the reſt, I choſe to inſert in places leſs ſuitable, rather than change the number of the propoſitions and the citations. I heartily beg that what I have here done may be read with candour, and that the defects I have been guilty of upon this difficult ſubject may be, not ſo much reprehended, as kindly ſupplied, and inveſtigated by new endeavours of my readers.*

Cambridge, Trin. Coll.
May 8. 1686.

IS. Newton.

In the ſecond Edition the ſecond Section of the firſt Book was enlarged. In the ſeventh Section of the ſecond Book the theory of the reſiſtances of fluids was more accurately inveſtigated, and confirmed by new experiments. In the third Book the Moon's Theory and the Præceſſion of the Æquinoxes were more fully deduced from their principles; and the theory of the Comets was confirmed by more examples of the calculation of their orbits, done alſo with greater accuracy.

In this third Edition, the reſiſtance of mediums is ſomewhat more largely handled than before; and new experiments of the reſiſtance of heavy bodies falling in air are added. In the third Book, the argument to prove that the Moon is retained in its orbit by the force of gravity is enlarged on. And there are added new obſervations of Mr. Pound'*s of the proportion of the diameters of* Jupiter *to each other: There are beſides added Mr.* Kirk's *obſervations of the Comet in* 1680. *the orbit of that Comet computed in an ellipſis by Dr.* Halley; *and the orbit of the Comet in* 1723. *computed by Mr.* Bradley.

THE PREFACE OF *Mr.* Roger Cotes,

To the Second Edition of this Work, ſo far as it relates to the Inventions and Diſcoveries herein contained.

THOSE who have treated of natural philoſophy, may be nearly reduced to three claſſes. Of theſe ſome have attributed to the ſeveral ſpecies of things, ſpecific and occult qualities; on which, in a manner unknown, they make the operations of the ſeveral bodies to depend. The ſum of the doctrine of the Schools derived from *Ariſtotle* and the Peripatetics is herein contained. They affirm that the ſeveral effects of bodies ariſe from the particular natures of thoſe bodies. But whence it is that bodies derive thoſe natures they don't tell us; and therefore they tell us nothing. And being entirely employed in giving names to things, and not in ſearching into things themſelves, we may ſay that they have invented a philoſophical way of ſpeaking, but not that they have made known to us true philoſophy.

Others therefore by laying aſide that uſeleſs heap of words, thought to employ their pains to better purpoſe. Theſe ſuppoſed all matter homogeneous, and that the variety of forms which is ſeen in bodies ariſes from ſome very plain and ſimple affections of the component particles. And by going on from ſimple things to thoſe which are more compounded they certainly proceed right; if they attribute no other properties to thoſe primary affections of the particles than Nature has done. But when they take a liberty of imagining at pleaſure unknown figures and magnitudes, and uncertain ſituations and motions of the parts; and moreover of ſuppoſing occult fluids, freely pervading the pores of bodies, endued with an all-performing ſubtilty, and agitated with occult motions; they now run out into dreams and chimera's, and neglect the true conſtitution of things; which certainly is not to be expected from fallacious conjectures, when we can ſcarce reach it by the moſt certain obſervations. Thoſe who fetch from hypotheſes the foundation on which they build their ſpeculations, may form indeed an ingenious romance, but a romance it will ſtill be.

There is left then the third claſs, which profeſs experimental philoſophy. Theſe indeed derive the cauſes of all things from the moſt ſimple principles poſſible;

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There is left then the third claſs, which profeſs experimental philoſophy. Theſe indeed derive the cauſes of all things from the moſt ſimple principles poſſible; but then they aſſume nothing as a principle, that is not proved by phænomena. They frame no hypotheſes, nor receive them into philoſophy otherwiſe than as queſtions whoſe truth may be diſputed. They proceed therefore in a twofold method, ſynthetical and analytical. From ſome ſelect phænomena they deduce by analyſis the forces of nature, and the more ſimple laws of forces; and from thence by ſyntheſis ſhew the conſtitution of the reſt. This is that incomparably beſt way of philoſophizing, which our renowned author moſt juſtly embraced before the reſt; and thought alone worthy to be cultivated and adorned by his excellent labours. Of this he has given us a moſt illuſtrious example, by the explication of the Syſtem of the World, moſt happily deduced from the Theory of Gravity. That the virtue of gravity was found in all bodies, others ſuſpected, or imagined before him; but he was the only and the firſt philoſopher that could demonſtrate it from appearances, and make it a ſolid foundation to the moſt noble ſpeculations.

I know indeed that ſome perſons and thoſe of great name, too much prepoſſeſſed with certain prejudices, are unwilling to aſſent to this new principle, and are ready to prefer uncertain notions to certain. It is not my intention to detract from the reputation of theſe eminent men; I ſhall only lay before the reader ſuch conſiderations as will enable him to paſs an equitable ſentence in this diſpute.

Therefore that we may begin our reaſoning from what is moſt ſimple and neareſt to us; let us conſider a little what is the nature of gravity with us on Earth, that we may proceed the more ſafely when we come to conſider it in the heavenly bodies, that lie at ſo vaſt a diſtance from us. It is now agreed by all philoſophers that all circumterreſtrial bodies gravitate towards the Earth. That no bodies really light are to be found, is now confirmed by manifold experience. That which is relative levity, is not true levity, but apparent only; and ariſes from the preponderating gravity of the contiguous bodies.

Moreover, as all bodies gravitate towards the Earth, ſo does the Earth again towards bodies. That the action of gravity is mutual, and equal on both ſides, is thus proved. Let the maſs of the Earth be diſtinguiſhed into any two parts whatever, either equal, or any how unequal; now if the weights of the parts towards each other were not mutually equal, the leſſer weight would give way to the greater, and the two parts joined together would move on ad infinitum in a right line towards that part to which the greater weight tends; altogether againſt experience. Therefore we muſt ſay that the weights of the parts are conſtituted in equilibrio; that is, that the action of gravity is mutual and equal on both ſides.

The weights of bodies, at equal diſtances from the centre of the Earth, are as the quantities of matter in the bodies. This is collected from the equal acceleration of all bodies that fall from a ſtate of reſt by the force of their weights; for the forces by which unequal bodies are equally accelerated muſt be proportional to the quantities of the matter to be moved. Now that all bodies are in falling equally accelerated appears from hence, that when the reſiſtance of the air is taken away, as it is under an exhauſted receiver, bodies falling de-

deſcribe equal ſpaces in equal times; and this is yet more accurately proved by the experiments of pendulums.

The attractive forces of bodies at equal diſtances, are as the quantities of matter in the bodies. For ſince bodies gravitate towards the Earth, and the Earth again towards bodies with equal moments; the weight of the Earth towards every body, or the force with which the body attracts the Earth, will be equal to the weight of the ſame body towards the Earth. But this weight was ſhewn to be as the quantity of matter in the body; and therefore the force with which every body attracts the Earth, or the abſolute force of the body, will be as the ſame quantity of matter.

Therefore the attractive force of the entire bodies ariſes from, and is compounded of, the attractive forces of the parts; becauſe as was juſt ſhewn, if the bulk of the matter be augmented or diminiſhed, its virtue is proportionably augmented or diminiſhed. We muſt therefore conclude that the action of the Earth is compounded of the united actions of its parts; and therefore that all terreſtrial bodies muſt attract each other mutually, with abſolute forces that are as the matter attracting. This is the nature of gravity upon Earth; let us now ſee what it is in the Heavens.

That every body perſeveres in its ſtate either of reſt, or of moving uniformly in a right line, unleſs in ſo far as it is compelled to change that ſtate by forces impreſſed, is a law of nature univerſally received by all philoſophers. But from thence it follows that bodies which move in curve lines, and are therefore continually going off from the right lines that are tangents to their orbits, are by ſome continued force retained in thoſe curvilinear paths. Since then the Planets move in curvilinear orbits there muſt be ſome force operating, by whoſe repeated actions they are perpetually made to deflect from the tangents.

Now it is collected by mathematical reaſoning, and evidently demonſtrated, that all bodies that move in any curve line deſcribed in a plane, and which, by a radius drawn to any point, whether quieſcent, or any how moved, deſcribe areas about that point proportional to the times, are urged by forces directed towards that point. This muſt therefore be granted. Since then all aſtronomers agree that the primary Planets deſcribe about the Sun, and the ſecondary about the primary, areas proportional to the times; it follows that the forces by which they are perpetually turned aſide from the rectilinear tangents, and made to revolve in curvilinear orbits, are directed towards the bodies that are ſituate in the centres of the orbits. This force may therefore not improperly be called centripetal in reſpect of the revolving body, and in reſpect of the central body attractive; whatever cauſe it may be imagined to ariſe from.

But beſides, theſe things muſt be alſo granted, as being mathematically demonſtrated: If ſeveral bodies revolve with an equable motion in concentric circles, and the ſquares of the periodic times are as the cubes of the diſtances from the common centre; the centripetal forces will be reciprocally as the ſquares of the diſtances. Or, if bodies revolve in orbits that are very near to circles, and the apſides of the orbits reſt; the centripetal forces of the revolving bodies will be reciprocally as the ſquares of the diſtances. That both theſe caſes hold in all the

Planets all aſtronomers conſent. Therefore the centripetal forces of all the Planets are reciprocally as the ſquares of the diſtances from the centres of their orbits. If any ſhould object, that the apſides of the Planets, and eſpecially of the Moon, are not perfectly at reſt; but are carried with a ſlow kind of motion in conſequentia; one may give this anſwer, that though we ſhould grant this very ſlow motion to ariſe from hence, that the proportion of the centripetal force is a little different from the duplicate, yet that we are able to compute mathematically the quantity of that aberration, and find it perfectly inſenſible. For the ratio of the Lunar centripetal force it ſelf, which muſt be the moſt irregular of them all, will be indeed a little greater than the duplicate, but will be near ſixty times nearer to that than it is to the triplicate. But we may give a truer anſwer, by ſaying that this progreſſion of the apſides ariſes not from an aberration from the duplicate proportion, but from a quite different cauſe, as is moſt admirably ſhewn in this philoſophy. It is certain then that the centripetal forces with which the primary Planets tend to the Sun, and the ſecondary to their primary, are accurately as the ſquares of the diſtances reciprocally.

From what has been hitherto ſaid, it is plain that the Planets are retained in their orbits by ſome force perpetually acting upon them; it is plain that that force is always directed towards the centres of their orbits; it is plain that its efficacy is augmented with the nearneſs to the centre, and diminiſhed with the ſame; and that it is augmented in the ſame proportion with which the ſquare of the diſtance is diminiſhed, and diminiſhed in the ſame proportion with which the ſquare of the diſtance is augmented. Let us now ſee whether, by making a compariſon between the centripetal forces of the Planets, and the ſorce of gravity, we may not by chance find them to be of the ſame kind. Now they will be of the ſame kind if we find on both ſides the ſame laws, and the ſame affections. Let us then firſt conſider the centripetal force of the Moon which is neareſt to us.

The rectilinear ſpaces, which bodies let fall from reſt deſcribe in a given time at the very beginning of the motion, when the bodies are urged by any forces whatſoever, are proportional to the ſorces. This appears from mathematical reaſoning. Thereſore the centripetal force of the Moon revolving in its orbit is to the force of gravity at the ſurſace of the Earth, as the ſpace, which in a very ſmall particle of time the Moon, deprived of all its circular force and deſcending by its centripetal force towards the Earth, would deſcribe, is to the ſpace which a heavy body would deſcribe, when falling by the force of its gravity near to the Earth, in the ſame given particle of time. The firſt of theſe ſpaces is equal to the verſed ſine of the arc deſcribed by the Moon in the ſame time, becauſe that verſed ſine meaſures the tranſlation of the Moon from the tangent, produced by the centripetal force; and therefore may be computed, if the periodic time of the Moon and its diſtance from the centre of the Earth are given. The laſt ſpace is found by experiments of pendulums, as Mr. *Huygens* has ſhewn. Therefore by making a calculation we ſhall find that the firſt ſpace is to the latter, or the centripetal force of the Moon revolving in its orbit will be to the force of gravity at the ſuperficies of the Earth, as the ſquare of the ſemi-diameter of the Earth to the ſquare of the ſemi-diameter of the orbit. But by what was ſhewn before the very ſame ratio holds between the centripetal force of the Moon revolving in its orbit, and the centripetal force of the Moon near the ſurface of the Earth. Therefore the centripetal force near the ſurface of the Earth is equal to the force of gravity. Therefore theſe are not two different forces, but one and the ſame; for if they were different, theſe forces united would cauſe bodies to deſcend to the Earth with twice the velocity they would fall with by the force of gravity alone. Therefore it is plain that the centripetal force, by which the Moon is perpetually, either impelled or attracted out of the tangent and retained in its orbit, is the very force of terreſtrial gravity reaching up to the Moon. And it is very reaſonable to believe that virtue ſhould extend it ſelf to vaſt diſtances, ſince upon the tops of the higheſt mountains we find no ſenſible diminution of it. Therefore the Moon gravitates towards the Earth; but on the other hand, the Earth by a mutual action equally gravitates towards the Moon; which is alſo abundantly confirmed in this philoſophy, where the Tides in the Sea and the Præceſſion of the Æquinoxes are treated of; which ariſe from the action both of the Moon and of the Sun upon the Earth. Hence laſtly, we diſcover by what law the force of gravity decreaſes at great diſtances from the Earth. For ſince gravity is no ways different from the Moon's centripetal force, and this is reciprocally proportional to the ſquare of the diſtance; it follows that it is in that very ratio that the force of gravity decreaſes.

Let us now go on to the reſt of the Planets. Becauſe the revolutions of the primary Planets about the Sun, and of the ſecondary about Jupiter and Saturn, are phænomena of the ſame kind with the revolution of the Moon about the Earth; and becauſe it has been moreover demonſtrated that the centripetal forces of the primary Planets are directed towards the centre of the Sun, and thoſe of the ſecondary towards the centres of Jupiter and Saturn, in the ſame manner as the centripetal force of the Moon is directed towards the centre of the Earth; and ſince beſides, all theſe forces are reciprocally as the ſquares of the diſtances from the centres, in the ſame manner as the centripetal force of the Moon is as the ſquare of the diſtance from the Earth; we muſt of courſe conclude, that the nature of all is the ſame. Therefore as the Moon gravitates towards the Earth, and the Earth again towards the Moon; ſo alſo all the ſecondary Planets, will gravitate towards their primary, and the primary Planets again towards their ſecondary; and ſo all the primary towards the Sun; and the Sun again towards the primary.

Therefore the Sun gravitates towards all the Planets, and all the Planets towards the Sun. For the ſecondary Planets, while they accompany the primary, revolve the mean while with the primary about the Sun. Therefore by the ſame argument, the Planets of both kinds gravitate towards the Sun; and the Sun towards them. That the ſecondary Planets gravitate towards the Sun is moreover abundantly clear from the inequalities of the Moon; a moſt accurate theory of which laid open with a moſt admirable ſagacity, we find explained in the third book of this work.

That the attractive virtue of the Sun is propagated on all ſides to prodigious diſtances, and is diffuſed to every part of the wide ſpace that ſurrounds it, is moſt evidently ſhewn by the motion of the Comets; which coming from places immenſely diſtant from the Sun, approach very near to it; and ſometimes ſo near, that in their perihelia they almoſt touch its body. The theory of theſe bodies was altogether unknown to aſtronomers, till in our own times our excellent author moſt happily diſcovered it, and demonſtrated the truth of it by moſt certain obſervations. So that it is now apparent that the Comets move in conic ſections having their foci in the Sun's centre, and by radij drawn to the Sun deſcribe areas proportional to the times. But from theſe phænomena it is manifeſt, and mathematically demonſtrated, that thoſe forces, by which the Comets are retained in their orbits, reſpect the Sun, and are reciprocally proportional to the ſquares of the diſtances from its centre. Therefore the Comets gravitate towards the Sun; and therefore the attractive force of the Sun not only acts on the bodies of the Planets, placed at given diſtances and very nearly in the ſame plane, but reaches alſo to the Comets in the moſt different parts of the heavens, and at the moſt different diſtances. This therefore is the nature of gravitating bodies, to propagate their force at all diſtances to all other gravitating bodies. But from thence it follows that all the Planets and Comets attract each other mutually, and gravitate mutually towards each other; which is alſo confirmed by the perturbation of Jupiter and Saturn, obſerved by aſtronomers, which is cauſed by the mutual actions of theſe two Planets upon each other;

as alſo from that very ſlow motion of the apſides, above taken notice of, and which ariſes from a like cauſe.

We have now proceeded ſo far as to ſhew that it muſt be acknowledged, that the Sun, and the Earth, and all the heavenly bodies attending the Sun, attract each other mutually. Therefore all the leaſt particles of matter in every one muſt have their ſeveral attractive forces, whoſe effect is as their quantity of matter; as was ſhewn above of the terreſtrial particles. At different diſtances theſe forces will be alſo in the duplicate ratio of the diſtances reciprocally; for it is mathematically demonſtrated, that particles attracting according to this law will compoſe globes attracting according to the ſame law.

The foregoing concluſions are grounded on this axiom which is received by all philoſophers; namely that effects of the ſame kind, that is, whoſe known properties are the ſame, take their riſe from the ſame cauſes and have the ſame unknown properties alſo. For who doubts, if gravity be the cauſe of the deſcent of a ſtone in *Europe*, but that it is alſo the cauſe of the ſame deſcent in *America?* If there is a mutual gravitation between a ſtone and the Earth in *Europe*, who will deny the ſame to be mutual in *America?* If in *Europe*, the attractive force of a ſtone and the Earth is compounded of the attractive forces of the parts; who will deny the like compoſition in *America?* If in *Europe*, the attraction of the Earth be propagated to all kinds of bodies and to all diſtances; why may it not as well be propagated in like manner in *America?* All philoſophy is founded on this rule; for if that be taken away we can affirm nothing of univerſals. The conſtitution of particular things is known by obſervations and experiments; and when that is done, it is by this rule that we judge univerſally of the nature of ſuch things in general.

Since then all bodies, whether upon Earth or in the Heavens, are heavy, ſo far as we can make any experiments or obſervations concerning them; we muſt certainly allow that gravity is found in all bodies univerſally. And in like manner as we ought not to ſuppoſe that any bodies can be otherwiſe than extended, moveable or impenetrable; ſo we ought not to conceive that any bodies can be otherwiſe than heavy. The extenſion, mobility and impenetrability of bodies become known to us only by experiments; and in the very ſame manner their gravity becomes known to us. All bodies we can make any obſervations upon, are extended, moveable and impenetrable; and thence we conclude all bodies, and thoſe we have no obſervations concerning, to be extended and moveable and impenetrable. So all bodies we can make obſervations on, we find to be heavy; and thence we conclude all bodies, and thoſe we have no obſervations of, to be heavy alſo. If any one ſhould ſay that the bodies of the fixed Stars are not heavy becauſe their gravity is not yet obſerved; they may ſay for the ſame reaſon that they are neither extended, nor moveable nor impenetrable, becauſe theſe affections of the fixed Stars are not yet obſerved. In ſhort, either gravity muſt have a place among the primary qualities of all bodies, or extenſion, mobility and impenetrability muſt not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extenſion, mobility and impenetrability.

Some I know diſapprove this concluſion, and mutter ſomething about occult qualities. They continually are cavilling with us, that gravity is an occult property; and occult cauſes are to be quite baniſhed from philoſophy. But to this the anſwer is eaſy; that thoſe are indeed occult cauſes whoſe exiſtence is occult; and imagined but not proved; but not thoſe whoſe real exiſtence is clearly demonſtrated by obſervations. Therefore gravity can by no means be called an occult cauſe of the celeſtial motions; becauſe it is plain from the phænomena that ſuch a virtue does really exiſt. Thoſe rather have recourſe to occult cauſes; who ſet imaginary vortices, of a matter entirely fictitious, and impreceptible by our ſenſes, to direct thoſe motions.

But ſhall gravity be therefore called an occult cauſe, and thrown out of philoſophy, becauſe the cauſe of gravity is occult and not yet diſcovered? Thoſe who affirm this, ſhould be careful not to fall into an abſurdity that may overturn the foundations of all philoſophy. For cauſes uſe to proceed in a continued chain from thoſe that are more compounded to thoſe that are more ſimple; when we are arrived at the moſt ſimple cauſe we can go no farther. Therefore no mechanical account or explanation of the moſt ſimple cauſe is to be expected or given; for if it could be given, the cauſe were not the moſt ſimple. Theſe moſt ſimple cauſes will you then call occult, and reject them? Then you muſt reject thoſe that immediately depend upon them, and thoſe which depend upon theſe laſt, till philoſophy is quite cleared and diſencumbred of all cauſes.

Some there are who ſay that gravity is præternatural, and call it a perpetual miracle. Therefore they would have it rejected, becauſe præternatural cauſes have no place in phyſics. It is hardly worth while to ſpend time in anſwering this ridiculous objection which overturns all philoſophy. For either they will deny gravity to be in bodies; which cannot be ſaid; or elſe, they will therefore call it præternatural becauſe it is not produced by the other affections of bodies, and therefore not by mechanical cauſes. But certainly there are primary affections of bodies; and theſe, becauſe they are primary, have no dependance on the others. Let them conſider whether all theſe are not in like manner præternatural, and in like manner to be rejected; and then what kind of philoſophy we are like to have.

Some there are who diſlike this celeſtial phyſics becauſe it contradicts the opinions of *Des Cartes*, and ſeems hardly to be reconciled with them. Let theſe enjoy their own opinion; but let them act fairly; and not deny the ſame liberty to us which they demand for themſelves. Since the *Newtonian* Philoſophy appears true to us, let us have the liberty to embrace and retain it, and to follow cauſes proved by phænomena, rather than cauſes only imagined, and not yet proved. The buſineſs of true philoſophy is to derive the natures of things from cauſes truly exiſtent; and to enquire after thoſe laws on which the Great Creator actually choſe to found this moſt beautiful Frame of the World; not thoſe by which he might have done the ſame, had he ſo pleaſed. It is reaſonable enough to ſuppoſe that from ſeveral cauſes, ſomewhat differing from each other, the ſame effect may ariſe; but the true cauſe will be that, from which it truly and actually does ariſe; the others have no place in true philoſophy. The ſame motion of the hour-hand in a clock may be occaſioned either by a weight hung, or a ſpring ſhut up within. But if a certain clock ſhould be really moved with a weight; we ſhould laugh at a man that would ſuppoſe it moved by a ſpring, and from that principle, ſuddenly taken up without farther examination, ſhould go about to explain the motion of the index; for certainly the way he ought to have taken ſhould have been, actually to look into the inward parts of the machine, that he might find the true principle of the propoſed motion. The like judgment ought to be made of thoſe philoſophers, who will have the heavens to be filled with a moſt ſubtile matter, which is perpetually carried round in vortices. For if they cou'd explain the phænomena ever ſo accurately by their hypotheſes, we could not yet ſay that they have diſcovered true philoſophy and the true cauſes of the celeſtial motions, unleſs they cou'd either demonſtrate that thoſe cauſes do actually exiſt, or at leaſt, that no others do exiſt. Therefore if it be made clear that the attraction of all bodies is a property actually exiſting *in rerum naturâ*; and if it be alſo ſhewn how the motions of the celeſtial bodies may be ſolved by that property; it would be very impertinent for any one to object, that theſe motions ought to be accounted for by vortices; even though we ſhould never ſo much allow ſuch an explication of thoſe motions to be poſſible. But we allow no ſuch thing; for the phænomena can by no means be accounted for by vortices;

tices; as our Author has abundantly proved from the cleareſt reaſons. So that Men muſt be ſtrangely fond of chimera's, who can ſpend their time ſo idly, as in patching up a ridiculous figment and ſetting it off with new comments of their own.

If the bodies of the Planets and Comets are carried round the Sun in vortices; the bodies ſo carried, and the parts of the vortices next ſurrounding them, muſt be carried with the ſame velocity and the ſame direction, and have the ſame denſity, and the ſame *vis inertiæ* anſwering to the bulk of the matter. But it is certain, the Planets and Comets, when in the very ſame parts of the Heavens, are carried with various velocities and various directions. Therefore it neceſſarily follows that thoſe parts of the celeſtial fluid, which are at the ſame diſtances from the Sun, muſt revolve at the ſame time with different velocities in different directions; for one kind of velocity and direction is required for the motion of the Planets, and another for that of the Comets. But ſince this cannot be accounted for; we muſt either ſay that all the celeſtial bodies are not carried about by vortices; or elſe that their motions are derived, not from one and the ſame vortex, but from ſeveral diſtinct ones, which fill and pervade the ſpaces round about the Sun.

But if ſeveral vortices are contained in the ſame ſpace, and are ſuppoſed to penetrate each other, and to revolve with different motions; then becauſe theſe motions muſt agree with thoſe of the bodies carried about by them, which are perfectly regular, and performed in conic ſections which are ſometimes very eccentric, and ſometimes nearly circles; one may very reaſonably ask, how it comes to paſs that theſe vortices remain entire, and have ſuffered no manner of perturbation in ſo many ages from the actions of the conflicting matter. Certainly if theſe fictitious motions are more compounded and more hard to be accounted for than the true motions of the Planets and Comets, it ſeems to no purpoſe to admit them into philoſophy; ſince every cauſe ought to be more ſimple than its effect. Allowing men to indulge their own fancies, ſuppoſe any man ſhould affirm that the Planets and Comets are ſurrounded with atmoſpheres like our Earth; which hypotheſis ſeems more reaſonable than that of vortices. Let him then affirm that theſe atmoſpheres by their own nature move about the Sun and deſcribe conic ſections, which motion is much more eaſily conceived than that of the vortices penetrating each other. Laſtly, that the Planets and Comets are carried about the Sun by theſe atmoſpheres of theirs; and then applaud his own ſagacity in diſcovering the cauſes of the celeſtial motions. He that rejects this fable muſt alſo reject the other; for two drops of water are not more like than this hypotheſis of atmoſpheres, and that of vortices.

Galileo has ſhewn, that when a ſtone projected moves in a parabola, its deflexion into that curve from its rectilinear path is occaſioned by the gravity of the ſtone towards the Earth, that is, by an occult quality. But now ſome body, more cunning than he, may come to explain the cauſe after this manner. He will ſuppoſe a certain ſubtile matter, not diſcernable by our ſight, our touch or any other of our ſenſes, which fills the ſpaces which are near and contiguous to the ſuperficies of the Earth; and that this matter is carried with different directions, and various, and often contrary, motions, deſcribing parabolic curves. Then ſee how eaſily he may account for the deflexion of the ſtone above ſpoken of. The ſtone, ſays he, floats in this ſubtile fluid, and following its motion, can't chuſe but deſcribe the ſame figure. But the fluid moves in parabolic curves; and therefore the ſtone muſt move in a parabola of courſe. Would not the acuteneſs of this philoſopher be thought very extraordinary, who could deduce the appearances of nature from mechanical cauſes, matter and motion, ſo clearly that the meaneſt man may underſtand it? Or indeed ſhould not we ſmile to ſee this new *Galileo* taking ſo much mathematical pains to introduce occult qualities into philoſophy, from whence they have been ſo happily excluded? But I am aſhamed to dwell ſo long upon trifles.

The ſum of the matter is this; the number of the Comets is certainly very great; their motions are perfectly regular; and obſerve the ſame laws with thoſe of the Planets. The orbits in which they move are conic ſections, and thoſe very eccentric. They move every way towards all parts of the Heavens, and paſs through the planetary regions with all poſſible freedom, and their motion is often contrary to the order of the ſigns. Theſe phænomena are moſt evidently confirmed by aſtronomical obſervations, and cannot be accounted for by vortices. Nay indeed they are utterly irreconcileable with the vortices of the Planets. There can be no room for the motions of the Comets; unleſs the celeſtial ſpaces be entirely cleared of that fictitious matter.

For if the Planets are carried about the Sun in vortices; the parts of the vortices which immediately ſurround every Planet muſt be of the ſame denſity with the Planet, as was ſhewn above. Thereſore all the matter contiguous to the perimeter of the *magnus orbis*, muſt be of the ſame denſity as the Earth. But now that which lies between the *magnus orbis* and the orb of Saturn muſt have either an equal or greater denſity. For to make the conſtitution of the vortex permanent, the parts of leſs denſity muſt lie near the centre, and thoſe of greater denſity muſt go farther from it. For ſince the periodic times of the Planets are in the ſeſquiplicate ratio of their diſtances from the Sun, the periods of the parts of the vortices muſt alſo preſerve the ſame ratio. Thence it will follow that the centrifugal forces of the parts of the vortex muſt be reciprocally as the ſquares of their diſtances. Thoſe parts therefore which are more remote from the centre endeavour to recede from it with leſs force; whence if their denſity be deficient, they muſt yield to the greater force with which the parts that lie nearer the centre endeavour to aſcend. Therefore the denſer parts will aſcend; and thoſe of leſs denſity will deſcend; and there will be a mutual change of places, till all the fluid matter in the whole vortex be ſo adjuſted and diſpoſed, that being reduced to an equilibrium its parts become quieſcent. If two fluids of different denſity be contained in the ſame veſſel; it will certainly come to paſs that the fluid of greater denſity will ſink the loweſt; and by a like reaſoning it follows that the denſer parts of the vortex by their greater centrifugal force will aſcend to the higheſt places. Therefore all that far greater part of the vortex which lies without the Earth's orb, will have a denſity, and by conſequence a *vis inertiæ* anſwering to the bulk of the matter, which cannot be leſs than the denſity and *vis inertiæ* of the Earth. But from hence will ariſe a mighty reſiſtance to the paſſage of the Comets, and ſuch as can't but be very ſenſible; not to ſay, enough to put a ſtop to, and abſorb, their motions entirely. But now it appears from the perfectly regular motion of the Comets, that they ſuffer no reſiſtance that is in the leaſt ſenſible; and therefore that they meet with no matter of any kind, that has any reſiſting force, or, by conſequence, any denſity or *vis inertiæ*. For the reſiſtance of mediums ariſes, either from the *inertia* of the matter of the fluid, or from its want of lubricity. That which ariſes from the want of lubricity is very ſmall, and is ſcarce obſervable in the fluids commonly known, unleſs they be very tenacious like oil and honey. The reſiſtance we find in air, water, quick-ſilver and the like fluids that are not tenacious, is almoſt all of the firſt kind; and cannot be diminiſhed by a greater degree of ſubtilty, if the denſity and *vis inertiæ*, to which this reſiſtance is proportional, remains; as is moſt evidently demonſtrated by our Author in his noble theory of reſiſtances in the ſecond book.

Bodies in going on through a fluid communicate their motion to the ambient fluid by little and little, and by that communication loſe their own motion, and by loſing it are retarded. Therefore the retardation is proportional to the motion communicated; and the communicated motion, when the velocity of the moving body is given, is as the denſity of the fluid; and therefore the retardation or reſiſtance will be as the ſame denſity of the fluid; nor can it be taken away, unleſs the fluid coming about to the hinder parts of the body reſtore the motion loſt. Now this cannot be done unleſs the impreſſion of the fluid on the hinder parts of the body be equal to the impreſſion of the fore parts of the body on the fluid, that is unleſs the relative velocity with which the fluid puſhes the body behind is equal to the velocity with which the body puſhes the fluid; that is, unleſs the abſolute velocity of the recurring fluid be twice as great as the abſolute velocity with which the fluid is driven forwards by the body; which is impoſſible. Therefore the reſiſtance of fluids ariſing from their *vis inertiæ* can by no means be taken away. So that we muſt conclude that the celeſtial fluid has no *vis inertiæ*, becauſe it has no reſiſting force; that it has no force to communicate motion with, becauſe it has no *vis inertiæ*; that it has no force to produce any change in one or more bodies, becauſe it has no force wherewith to communicate motion; that it has no manner of efficacy, becauſe it has no faculty wherewith to produce any change of any kind. Therefore certainly this hypotheſis may be juſtly called ridiculous, and unworthy a philoſopher; ſince it is altogether without foundation, and does not in the leaſt ſerve to explain the nature of things. Thoſe who would have the Heavens filled with a fluid matter, but ſuppoſe it void of any *vis inertiæ*; do indeed in words deny a vacuum, but allow it in fact. For ſince a fluid matter of that kind can no ways be diſtinguiſhed from empty ſpace; the diſpute is now about the names, and not the natures of things. If any are ſo fond of matter, that they will by no means admit of a ſpace void of body; let us conſider, where they muſt come at laſt.

For either they will ſay, that this conſtitution of a world every where full, was made ſo by the will of GOD to this end, that the operations of

Nature might be aſſiſted every where by a ſubtile æther pervading and filling all things; which cannot be ſaid however, ſince we have ſhewn from the phænomena of the Comets, that this æther is of no efficacy at all; or they will ſay, that it became ſo by the ſame will of GOD for ſome unknown end; which ought not to be ſaid, becauſe for the ſame reaſon a different conſtitution may be as well ſuppoſed; or laſtly, they will not ſay that it was cauſed by the will of GOD, but by ſome neceſſity of its nature. Therefore they will at laſt ſink into the mire of that infamous herd; who dream that all things are governed by Fate, and not by Providence; and that matter exiſts by the neceſſity of its nature always and every where, being infinite and eternal. But ſuppoſing theſe things, it muſt be alſo every where uniform; for variety of forms is entirely inconſiſtent with neceſſity. It muſt be alſo unmoved; for if it be neceſſarily moved in any determinate direction, with any determinate velocity, it will by a like neceſſity be moved in a different direction with a different velocity; but it can never move in different directions with different velocities; therefore it muſt be unmoved. Without all doubt this World, ſo diverſified with that variety of forms and motions we find in it, could ariſe from nothing but the perfectly free will of GOD directing and preſiding over all.

From this fountain it is that thoſe laws, which we call the laws of Nature, have flowed; in which there appear many traces indeed of the moſt wiſe contrivance, but not the leaſt ſhadow of neceſſity. Theſe therefore we muſt not ſeek from uncertain conjectures, but learn them from obſervations and experiments.

periments. He who thinks to find the true principles of phyſics and the laws of natural things by the force alone of his own mind, and the internal light of his reaſon; muſt either ſuppoſe that the World exiſts by neceſſity, and by the ſame neceſſity follows the laws propoſed; or if the order of Nature was eſtabliſhed by the will of GOD, that himſelf, a miſerable reptile, can tell what was fitteſt to be done. All ſound and true philoſophy is founded on the appearances of things; which if they draw us never ſo much againſt our wills, to ſuch principles as moſt clearly manifeſt to us the moſt excellent counſel and ſupreme dominion of the All-wiſe and Almighty Being; thoſe principles are not therefore to be laid aſide, becauſe ſome men may perhaps diſlike them. They may call them, if they pleaſe, miracles or occult qualities; but names maliciouſly given ought not to be a diſadvantage to the things themſelves; unleſs they will ſay at laſt, that all philoſophy ought to be founded in atheiſm. Philoſophy muſt not be corrupted in complaiſance to theſe men; for the order of things will not be changed.

Fair and equal judges will therefore give ſentence in favour of this moſt excellent method of philoſophy, which is founded on experiments and obſervations. To this method it is hardly to be ſaid or imagined, what light, what ſplendor, hath accrued from this admirable work of our illuſtrious author; whoſe happy and ſublime genius, reſolving the moſt difficult problems, and reaching to diſcoveries of which the mind of man was thought incapable before, is deſervedly admired by all thoſe who are ſomewhat more than ſuperficially verſed in theſe matters. The gates are now ſet open; and by his means we may freely enter into the knowledge of the hidden ſecrets and wonders of natural things. He has ſo clearly laid open and ſet before our eyes the moſt beautiful frame of the Syſtem of the World; that if King *Alphonſus* were now alive, he would not complain for want of the graces either of ſimplicity or of harmony in it. Therefore we may now more nearly behold the beauties of Nature, and entertain our ſelves with the delightful contemplation; and, which is the beſt and moſt valuable fruit of philoſophy, be thence incited the more profoundly to reverence and adore the great Maker and Lord of all. He muſt be blind who from the moſt wiſe and excellent contrivances of things cannot ſee the infinite Wiſdom and Goodneſs of their Almighty Creator, and he muſt be mad and ſenſeleſs, who refuſes to acknowledge them.

THE MATHEMATICAL PRINCIPLES OF *Natural Philoſophy.*

DEFINITIONS.

DEF. I.

The Quantity of Matter is the meaſure of the ſame, ariſing from its denſity and bulk conjunctly.

THUS air of a double denſity, in a double ſpace, is quadruple in quantity; in a triple ſpace, ſextuple in quantity. The ſame thing is to be underſtood of ſnow, and fine duſt or powders, that are condenſed by compreſſion or liquefaction; and of all bodies that are by any cauſes whatever differently condenſed. I have no regard in this place to a medium, if any ſuch there is, that freely pervades the interſtices between the parts of bodies. It is this quantity that I mean hereafter every where under the name of Body or Maſs. And the ſame is known by the weight of each body: For it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which ſhall be ſhewn hereafter.

DEFINITION II.

The Quantity of Motion is the meaſure of the ſame, ariſing from the velocity and quantity of matter conjunctly.

The motion of the whole is the Sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

DEFINITION III.

The Vis Inſita, *or Innate Force of Matter, is a power of reſiſting, by which every body, as much as in it lies, endeavours to perſevere in its preſent ſtate, whether it be of reſt, or of moving uniformly forward in a right line.*

This force is ever proportional to the body whoſe force it is; and differs nothing from the inactivity of the Maſs, but in our manner of conceiving it. A body from the inactivity of matter, is not without difficulty put out of its ſtate of reſt or motion. Upon which account, this *Vis inſita*, may, by a moſt ſignificant name, be called *Vis Inertiæ*, or Force of Inactivity. But a body exerts this force only, when another force impreſs'd upon it, endeavours to change its condition; and the exerciſe of this force may be conſidered both as reſiſtance and impulſe: It is reſiſtance in ſo far as the body, for maintaining its preſent ſtate withſtands the force impreſſed; it is impulſe, in ſo far as the body, by not eaſily giving way to the impreſs'd force of another, endeavours to change the ſtate of that other. Reſiſtance is uſually aſcrib'd to bodies at reſt, and impulſe to thoſe in motion: But motion and reſt, as commonly conceived, are only relatively diſtinguiſhed; nor are thoſe bodies always truly at reſt, which commonly are taken to be ſo.

DEFINITION IV.

An impreſs'd force is an action exerted upon a body, in order to change its ſtate, either of reſt, or of moving uniformly forward in a right line.

This force conſiſts in the action only; and remains no longer in the body, when the action is over. For a body maintains every new ſtate it acquires, by its *Vis Inertiæ* only. Impreſs'd forces are of different origines; as from percuſſion, from preſſure, from centripetal force.

DEFINITION V.

A Centripetal force is that by which bodies are drawn or impell'd, or any way tend, towards a point as to a centre.

Of this ſort is Gravity by which bodies tend to the centre of the Earth; Magnetiſm, by which iron tends to the loadſtone; and that force, whatever it is, by which the Planets are perpetually drawn aſide from the rectilinear motions, which otherwiſe they wou'd purſue, and made to revolve in curvilinear orbits. A ſtone, whirled about in a ſling, endeavours to recede from the hand that turns it; and by that endeavour, diſtends the ſling, and that with ſo much the greater force, as it is revolv'd with the greater velocity; and as ſoon as ever it is let go, flies away. That force which oppoſes it ſelf to this endeavour, and by which the ſling perpetually draws back the ſtone towards the hand, and retains it in its orbit, becauſe 'tis directed to the hand as the centre of the orbit, I call the Centripetal force. And the ſame thing is to be underſtood of all bodies, revolv'd in any orbits. They all endeavour to recede from the centres of their orbits; and were it not for the oppoſition of a contrary force which reſtrains them to, and detains them in their orbits, which I therefore call Centripetal, would fly off in right lines, with an uniform motion. A projectile, if it was not for the force of gravity, would not deviate towards the Earth, but would go off from it in a right line and that with an uniform motion, if the reſiſtance of the Air was taken away. 'Tis by its gravity that it is drawn aſide perpetually from its rectilinear courſe, and made to deviate towards the Earth, more or leſs, according to the force of its gravity, and the velocity of its motion. The leſs its gravity is, for the quantity of its matter, or the greater the velocity with which it is projected, the leſs will it deviate from a rectilinear courſe, and the farther it will go. If a leaden ball projected from the top of a mountain by the force of gun-powder with a given velocity, and in a direction parallel to the horizon, is carried in a curve line to the diſtance of two miles before it falls to the ground; the ſame, if the reſiſtance of the Air was took away, with a double or decuple velocity, would fly twice or ten times as far. And by increaſing the velocity, we may at pleaſure increaſe the diſtance to which it might be projected, and diminiſh the curvature of the line, which it might deſcribe, till at laſt it ſhould fall at the diſtance of 10, 30, or 90 degrees, or even might go quite round the whole Earth before it falls; or laſtly, ſo that it might never fall to the Earth, but go forwards into the Celeſtial Spaces, and proceed in its motion *in infinitum*. And after the ſame manner, that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole Earth, the Moon alſo, either by the force of gravity, if it is endued with gravity, or by any other force, that impells it towards the Earth, may be perpetually drawn aſide towards the Earth, out of the rectilinear way, which by its innate force it would purſue; and be made to revolve in the orbit which it now deſcribes: nor could the Moon without ſome ſuch force, be retain'd in its orbit. If this force was too ſmall, it would not ſufficiently turn the Moon out of a rectilinear courſe: if it was too great, it would turn it too much, and draw down the Moon from its orbit towards the Earth. It is neceſſary, that the force be of a juſt quantity, and it belongs to the Mathematicians to find the force, that may ſerve exactly to retain a body in a given orbit, with a given velocity; and *vice verſa,* to determine the curvilinear way, into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force.

The quantity of any Centripetal Force may be conſidered as of three kinds, Abſolute, Accelerative, and Motive.

DEFINITION VI.

The abſolute quantity of a centripetal force is the meaſure of the ſame, proportional to the efficacy of the cauſe that propagates it from the centre, through the ſpaces round about.

Thus the magnetic force is greater in one load-ſtone and leſs in another, according to their ſizes and ſtrength.

DEFINITION VII.

The accelerative quantity of a centripetal force is the meaſure of the ſame, proportional to the velocity which it generates in a given time.

Thus the force of the ſame load-ſtone is greater at a leſs diſtance, and leſs at a greater: alſo the force of gravity is greater in valleys, leſs on tops of exceeding high mountains; and yet leſs (as ſhall be hereafter ſhewn) at greater diſtances from the body of the

Earth; but at equal diſtances, it is the ſame every where; becauſe (taking away, or allowing for, the reſiſtance of the Air) it equally accelerates all falling bodies, whether heavy or light, great or ſmall.

DEFINITION VIII.

The motive quantity of a centripetal force, is the meaſure of the ſame, proportional to the motion which it generates in a given time.

Thus the Weight is greater in a greater body, leſs in a leſs body; it is greater near to the Earth, and leſs at remoter diſtances. This ſort of quantity is the centripetency, or propenſion of the whole body towards the centre, or as I may ſay, its Weight; and it is ever known by the quantity of a force equal and contrary to it, that is juſt ſufficient to hinder the deſcent of the body.

Theſe quantities of Forces, we may for brevity's ſake call by the names of Motive, Accelerative and Abſolute forces; and for diſtinction ſake conſider them, with reſpect to the Bodies that tend to the centre; to the Places of thoſe bodies; and to the Centre of force towards which they tend: That is to ſay, I refer the Motive force to the Body, as an endeavour and propenſity of the whole towards a centre, ariſing from the propenſities of the ſeveral parts taken together; the Accelerative force to the Place of the body, as a certain power or energy diffuſed from the centre to all places around to move the bodies that are in them; and the Abſolute force to the Centre, as indued with ſome cauſe, without which thoſe motive forces would not be propagated through the ſpaces round about; whether that cauſe is ſome central body, (ſuch as is the Load-ſtone, in the centre of the force of Magnetiſm, or the Earth in the centre of the gravitating force) or any thing elſe that does not yet appear. For I here deſign only to give a Mathematical notion of thoſe forces, without conſidering their Phyſical cauſes and ſeats.

Wherefore the Accelerative force will ſtand in the ſame relation to the Motive, as celerity does to motion. For the quantity of motion ariſes from the celerity, drawn into the quantity of matter; and the motive force ariſes from the accelerative force drawn into the ſame quantity of matter. For the ſum of the actions of the Accelerative force, upon the ſeveral particles of the body, is the Motive force of the whole. Hence it is, that near the ſurface of the Earth, where the accelerative gravity, or force productive of gravity in all bodies is the ſame, the motive gravity or the Weight is as the Body: but if we ſhould aſcend to higher regions, where the accelerative gravity is leſs, the Weight would be likewiſe diminiſhed, and would always be as the product of the Body, by the Accelerative gravity. So in thoſe regions, where the accelerative gravity is diminiſhed into one half, the Weight of a body two or three times leſs, will be four or ſix times leſs.

I likewiſe call Attractions and Impulſes, in the ſame ſenſe, Accelerative, and Motive; and uſe the words Attraction, Impulſe or Propenſity of any ſort towards a centre, promiſcuouſly, and indifferently, one for another; conſidering thoſe forces not Phyſically but Mathematically: Wherefore, the reader is not to imagine, that by thoſe words, I any where take upon me to define the kind, or the manner of any Action, the cauſes or the phyſical reaſon thereof, or that I attribute Forces, in a true and Phyſical ſenſe, to certain centres (which are only Mathematical points); when at any time I happen to ſpeak of centres as attracting, or as endued with attractive powers.

SCHOLIUM.

Hitherto I have laid down the definitions of ſuch words as are leſs known, and explained the ſenſe in which I would have them to be underſtood in the following diſcourſe. I do not define Time, Space, Place and Motion, as being well known to all. Only I muſt obſerve, that the vulgar conceive thoſe quantities under no other notions but from the relation they bear to ſenſible objects. And thence ariſe certain prejudices, for the removing of which, it will be convenient to diſtinguiſh them into Abſolute and Relative, True and Apparent, Mathematical and Common.

I. Abſolute, True, and Mathematical Time, of it ſelf, and from its own nature flows equably without regard to any thing external, and by another name is called Duration: Relative, Apparent, and Common Time is ſome ſenſible and external (whether accurate or unequable) meaſure of Duration by the means of motion, which is commonly uſed inſtead of True time; ſuch as an Hour, a Day, a Month, a Year.

II Abſolute Space, in its own nature, without regard to any thing external, remains always ſimilar and immoveable. Relative Space is ſome moveable dimenſion or meaſure of the abſolute ſpaces; which our ſenſes determine, by its poſition to bodies; and which is vulgarly taken for immoveable ſpace; Such is the dimenſion of a ſubterraneous, an aereal, or celeſtial ſpace, determin'd by its poſition in reſpect of the Earth. Abſolute and Relative ſpace, are the ſame in figure and magnitude; but they do not remain always numerically the ſame. For if the Earth, for inſtance, moves; a ſpace a ſpace of our Air, which relatively and in reſpect of the Earth, remains always the ſame, will at one time be one part of the abſolute ſpace into which the Air paſſes; at another time it will be another part of the ſame, and ſo, abſolutely underſtood, it will be perpetually mutable.

III. Place is a part of ſpace which a body takes up, and, is according to the ſpace, either abſolute or relative. I ſay, a Part of Space; not the ſituation, nor the external ſurface of the body. For the places of equal Solids, are always equal; but their ſuperficies, by reaſon of their diſſimilar figures, are often unequal. Poſitions properly have no quantity, nor are they ſo much the places themſelves, as the properties of places. The motion of the whole is the ſame thing with the ſum of the motions of the parts, that is, the tranſlation of the whole, out of its place, is the ſame thing with the ſum of the tranſlations of the parts out of their places; and therefore the Place of the whole, is the ſame thing with the Sum of the places of the parts, and for that reaſon, it is internal, and in the whole body.

IV. Abſolute motion, is the tranſlation of a body from one abſolute place into another; and Relative motion, the tranſlation from one relative place into another. Thus in a Ship under ſail, the relative place of a body is that part of the Ship, which the Body poſſeſſes; or that part of its cavity which the body fills, and which therefore moves together with the Ship: And Relative reſt, is the continuance of the Body in the ſame part of the Ship, or of its cavity. But Real, abſolute reſt, is the continuance of the Body in the ſame part of that Immoveable ſpace, in which the Ship it ſelf, its cavity, and all that it contains, is moved. Wherefore, if the Earth is really at reſt, the Body, which relatively reſts in the Ship, will really and abſolutely move with the ſame velocity which the Ship has on the Earth. But if the Earth alſo moves, the true and abſolute motion of the body will ariſe, partly from the true motion of the Earth, in immoveable ſpace; partly from the relative motion of the Ship on the Earth: and if the body moves alſo relatively in the Ship; its true motion will ariſe, partly from the true motion of the Earth, in immoveable ſpace, and partly from the relative motions as well of the Ship on the Earth, as of the Body in the Ship; and from theſe relative motions, will ariſe the relative motion of the Body on the Earth. As if that part of the Earth where the Ship is, was truly mov'd toward the Eaſt, with a velocity of 10010 parts; while the Ship it ſelf with a freſh gale, and full ſails, is carry'd towards the Weſt, with a velocity expreſs'd by 10 of thoſe parts; but a Sailor walks in the Ship towards the Eaſt, with 1 part of the ſaid velocity: then the Sailor will be moved truly and abſolutely in immoveable ſpace towards the Eaſt with a velocity of 10001 parts, and relatively on the Earth towards the Weſt, with a velocity of 9 of thoſe parts.

Abſolute time, in Aſtronomy, is diſtinguiſh'd from Relative, by the Equation or correction of the vulgar time. For the natural days are truly unequal, though they are commonly conſider'd as equal, and uſed for a meaſure of time: Aſtronomers correct this inequality for their more accurate deducing of the celeſtial motions It may be, that there is no ſuch thing as an equable motion, whereby time may be accurately meaſured. All motions may be accelerated and retarded, but the True, or equable progreſs, of Abſolute time is liable to no change. The duration or perſeverance of the exiſtence of things remains the ſame, whether the motions a ſpace of our Air, which relatively and in reſpect of the Earth, remains always the ſame, will at one time be one part of the abſolute ſpace into which the Air paſſes; at another time it will be another part of the ſame, and ſo, abſolutely underſtood, it will be perpetually mutable.

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As the order of the parts of Time is immutable, ſo alſo is the order of the parts of Space. Suppoſe thoſe parts to be mov'd out of their places, and they will be moved (if the expreſſion may be allowed) out of themſelves. For times and ſpaces are, as it were, the Places as well of themſelves as of all other things. All things are placed in Time as to order of Succeſſion; and in Space as to order of Situation. It is from their eſſence or nature that they are Places; and that the primary places of things ſhould be moveable, is abſurd. Theſe are therefore the abſolute places; and tranſlations out of thoſe places, are the only Abſolute Motions.

But becauſe the parts of Space cannot be ſeen, or diſtinguiſhed from one another by our Senſes, therefore in their ſtead we uſe ſenſible meaſures of them. For from the poſitions and diſtances of things from any body conſider'd as immoveable, we define all places: and then with reſpect to ſuch places, we eſtimate all motions, conſidering bodies as transfer'd from ſome of thoſe places into others. And ſo inſtead of abſolute places and motions, we uſe relative ones; and that without any inconvenience in common affairs: but in Philoſophical diſquiſitions, we ought to abſtract from our ſenſes, and conſider things themſelves, diſtinct from what are only ſenſible meaſures of them. For it may be that there is no body really at reſt, to which the places and motions of others may be referr'd.

But we may diſtinguiſh Reſt and Motion, abſolute and relative, one from the other by their Properties, Cauſes and Effects. It is a property of Reſt, that bodies really at reſt do reſt in reſpect of one another. And therefore as it is poſſible, that in the remote regions of the fixed Stars, or perhaps far beyond them, there may be ſome body abſolutely at reſt; but impoſſible to know from the poſition of bodies to one another in our regions, whether any of theſe do keep the ſame poſition to that remote body; it follows that abſolute reſt cannot be determined from the poſition of bodies in our regions.

It is a property of motion, that the parts, which retain given poſitions to their wholes, do partake of the motions of thoſe wholes. For all the parts of revolving bodies endeavour to recede from the axe of motion; and the impetus of bodies moving forwards, ariſes from the joint impetus of all the parts. Therefore, if ſurrounding bodies are mov'd, thoſe that are relatively at reſt within them, will partake of their motion. Upon which account, the true and abſolute motion of a body cannot be determin'd by the tranſlation of it from thoſe which only ſeem to reſt: For the external bodies ought not only to appear at reſt, but to be really at reſt. For otherwiſe, all included bodies, beſide their tranſlation from near the ſurrounding ones, partake likewiſe of their true motions; and tho' that tranſlation was not made they would not be really at reſt, but only ſeem to be ſo. For the ſurrounding bodies ſtand in the like relation to the ſurrounded, as the exteriour part of a whole does to the interiour, or as the ſhell does to the kernel; but, if the ſhell moves, the kernel will alſo move, as being part of the whole, without any removal from near the ſhell.

A property near a kin to the preceding, is this, that if a place is mov'd, whatever is placed therein moves along with it; and therefore a body, which is mov'd from a place in motion, partakes alſo of the motion of its place. Upon which account all motions from places in motion, are no other than parts of entire and abſolute motions; and every entire motion is compoſed out of the motion of the body out of its firſt place, and the motion of this place out of its place, and ſo on; until we come to ſome immoveable place, as in the before mention'd example of the Sailor. Wherefore entire and abſolute motions can be no otherwiſe determin'd than by immoveable places; and for that reaſon I did before refer thoſe abſolute motions to immoveable places, but relative ones to moveable places. Now no other places are immoveable, but thoſe that, from infinity to infinity, do all retain the ſame given poſitions one to another; and upon this account, muſt ever remain unmov'd; and do thereby conſtitute, what I call, immoveable ſpace.

The Cauſes by which true and relative motions are diſtinguiſhed, one from the other, are the forces impreſs'd upon bodies to generate motion. True motion is neither generated nor alter'd, but by ſome force impreſs'd upon the body moved: but relative motion may be generated or alter'd without any force impreſs'd upon the body. For it is ſufficient only to impreſs ſome force on other bodies with which the former is compar'd, that by their giving way, that relation may be chang'd, in which the relative reſt or motion of this other body did conſiſt. Again, True motion ſuffers always ſome change from any force impreſs'd upon the moving body; but Relative motion does not neceſſarily undergo any change, by ſuch forces. For if the ſame forces are likewiſe impreſs'd on thoſe other bodies, with which the compariſon is made, that the relative poſition may be preſerved, then that condition will be preſerv'd, in which the relative motion conſiſts. And therefore, any relative motion may be changed, when the true motion remains unalter'd, and the relative may be preſerv'd, when the true ſuffers ſome change. Upon which accounts, true motion does by no means conſiſt in ſuch relations.

The Effects which diſtinguiſh abſolute from relative motion are, the forces of receding from the axe of circular motion. For there are no ſuch forces in a circular motion purely relative, but in a true and abſolute circular motion, they are greater or leſs, according to the quantity of the motion. If a veſſel, hung by a long cord, is ſo often turned about that the cord is ſtrongly twiſted, then fill'd with water, and held at reſt together with the water; after by the ſudden action of another force, it is whirl'd about the contrary way, and while the cord is untwiſting it ſelf, the veſſel continues for ſome time in this motion; the ſurface of the water will at firſt be plain, as before the veſſel began to move: but the veſſel, by gradually communicating its motion to the water, will make it begin ſenſibly to revolve, and recede by little and little from the middle, and aſcend to the ſides of the veſſel, forming it ſelf into a concave figure, (as I have experienced) and the ſwifter the motion becomes, the higher will the water riſe, till at laſt, performing its revolutions in the ſame times with the veſſel, it becomes relatively at reſt in it. This aſcent of the water ſhews its endeavour to recede from the axe of its motion; and the true and abſolute circular motion of the water, which is here directly contrary to the relative, diſcovers it ſelf, and may be meaſured by this endeavour. At firſt, when the relative motion of the water in the veſſel was greateſt it produc'd no endeavour to recede from the axe: the water ſhew'd no tendency to the circumference, nor any aſcent towards the ſides of the veſſel, but remain'd of a plain ſurface, and therefore its True circular motion had not yet begun. But afterwards, when the relative motion of the water had decreas'd, the aſcent thereof towards the ſides of the veſſel, prov'd its endeavour to recede from the axe; and this endeavour ſhew'd the real circular motion of the water perpetually increaſing, till it had acquir'd its greateſt quantity, when the water reſted relatively in the veſſel. And therefore this endeavour does not depend upon any tranſlation of the water in reſpect of the ambient bodies, nor can true circular motion be defin'd by ſuch tranſlations. There is only one real circular motion of any one revolving body, correſponding to only one power of endeavouring to recede from its axe of motion, as its proper and adequate effect: but relative motions in one and the ſame body are innumerable, according to the various relations it bears to external bodies, and like other relations, are altogether deſtitute of any real effect, any otherwiſe than they may perhaps participate of that one only true motion. And therefore in their ſyſtem who ſuppoſe that our heavens, revolving below the ſphere of the fixt Stars, carry the Planets along with them; the ſeveral parts of thoſe heavens, and the Planets, which are indeed relatively at reſt in their heavens, do yet really move. For they change their poſition one to another (which never happens to bodies truly at reſt) and being carried together with their heavens, participate of their motions, and as parts of revolving wholes, endeavour to recede from the axe of their motions.

Wherefore relative quantities, are not the quantities themſelves, whoſe names they bear, but thoſe ſenſible meaſures of them (either accurate or inaccurate) which are commonly uſed inſtead of the meaſur'd quantities themſelves. And if the meaning of words is to be determin'd by their uſe; then by the names Time, Space, Place and Motion, their meaſures are properly to be underſtood; and the expreſſion will be unuſual, and purely Mathematical, if the meaſured quantities themſelves are meant. Upon which account, they do ſtrain the Sacred Writings, who there interpret thoſe words for the meaſur'd quantities. Nor do thoſe leſs defile the purity of Mathematical and Philoſophical truths, who confound real quantities themſelves with their relations and vulgar meaſures.

It is indeed a matter of great difficulty to diſcover, and effectually to diſtinguiſh, the True motions of particular bodies from the Apparent: becauſe the parts of that immoveable ſpace in which thoſe motions are perform'd, do by no means come under the obſervation of our ſenſes. Yet the thing is not altogether deſperate; for we have ſome arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the cauſes and effects of the true motions. For inſtance, if two globes kept at a given diſtance one from the other, by means of a cord that connects them, were revolv'd about their common centre of gravity; we might, from the tenſion of the cord, diſcover the endeavour of the globes to recede from the axe of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces ſhould be impreſs'd at once on the alternate faces of the globes to augment or diminiſh their circular motions; from the encreaſe or decreaſe of the tenſion of the cord, we might infer the increment or decrement of their motions; and thence would be found, on what faces thoſe forces ought to be impreſs'd, that the motions of the globes might be moſt augmented, that is, we might diſcover their hindermoſt faces, or thoſe which, in the circular motion, do follow. But the faces which follow being known, and conſequently, the oppoſite ones that precede, we ſhould likewiſe know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, ev'n in an immenſe vacuum, where there was nothing external or ſenſible with which the globes could be compar'd. But now if in that ſpace ſome remote bodies were plac'd that kept always a given poſition one to another, as the Fixt Stars do in our regions; we cou'd not indeed determine from the relative tranſlation of the globes among thoſe bodies, whether the motion did belong to the globes or to the bodies. But if we obſerv'd the cord, and found that its tenſion was that very tenſion which the motions of the globes requir'd, we might conclude the motion to be in the globes, and the bodies to be at reſt; and then, laſtly, from the tranſlation of the globes among the bodies, we ſhould find the determination of their motions. But how we are to collect the true motions from their cauſes, effects, and apparent differences; and *vice verſa*, how from the motions, either true or apparent, we may come to the knowledge of their cauſes and effects, ſhall be explain'd more at large in the following Tract. For to this end it was that I compos'd it.

Axioms or Laws of Motion.

LAW I.

Every body perſeveres in its ſtate of reſt, or of uniform motion in a right line, unleſs it is compelled to change that ſtate by forces impreſs'd thereon.

PRojectiles perſevere in their motions, ſo far as they are not retarded by the reſiſtance of the air, or impell'd downwards by the force of gravity. A top, whoſe parts by their coheſion are perpetually drawn aſide from rectilinear motions, does not ceaſe its rotation, otherwiſe than as it is retarded by the air. The greater bodies of the Planets and Comets, meeting with leſs reſiſtance in more free ſpaces, preſerve their motions both progreſſive and circular for a much longer time.

LAW II.

The alteration of motion is ever proportional to the motive force impreſs'd; and is made in the direction of the right line in which that force is impreſs'd.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impreſs'd altogether and at once, or gradually and ſucceſſively. And this motion (being always directed the ſame way with the generating force) if the body moved before, is added to or ſubducted from the former motion, according as they directly conſpire with or are directly contrary to each other; or obliquely joyned, when they are oblique, ſo as to produce a new motion compounded from the determination of both.

LAW III.

To every Action there is always oppoſed an equal Reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or preſſes another is as much drawn or preſſed by that other. If you preſs a ſtone with your finger, the finger is alſo preſſed by the ſtone. If a horſe draws a ſtone tyed to a rope, the horſe (if I may ſo ſay) will be equally drawn back towards the ſtone: For the diſtended rope, by the ſame endeavour to relax or unbend it ſelf, will draw the horſe as much towards the ſtone, as it does the ſtone towards the horſe, and will obſtruct the progreſs of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other; that body alſo (becauſe of the equality of the mutual preſſure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by theſe actions are equal, not in the velocities, but in the motions of bodies; that is to ſay, if the bodies are not hinder'd by any other impediments. For becauſe the motions are equally changed, the changes of the velocities made towards contrary parts, are reciprocally proportional to the bodies. This Law takes place alſo in Attractions, as will be proved in the next Scholium.

COROLLARY I.

A body by two forces conjoined will deſcribe the diagonal of a parallelogram, in the ſame time that it would deſcribe the ſides, by thoſe forces apart. (Pl. 1. Fig. 1.)

If a body in a given time, by the force *M* impreſs'd apart in the place *A*, ſhould with an uniform motion be carried from *A* to *B*; and by the force *N* impreſs'd apart in the ſame place, ſhould be carried from *A* to *C*: compleat the parallelogram *ABCD*, and by both forces acting together, it will in the ſame time be carried in the diagonal from *A* to *D*. For ſince the force *N* acts in the direction of the line *AC*, parallel to *BD*, this force (by the ſecond law) will not at all alter the velocity generated by the other force *M*, by which the body is carried towards the line *BD*. The body therefore will arrive at the line *BD* in the ſame time, whether the force *N* be impreſs'd or not; and therefore at the end of that time, it will be found ſomewhere in the line *BD*. By the ſame argument, at the end of the ſame time it will be found ſomewhere in the line *CD*. Therefore it will be found in the point *D*, where both lines meet. But it will move in a right line from *A* to *D* by Law 1.

COROLLARY II.

And hence is explained the compoſition of any one direct force A D, *out of any two oblique forces* A B *and* B D; *and, on the contrary the reſolution of any one direct force* A D *into two oblique forces* A B *and* B D: *which compoſition and reſolution are abundantly confirmed from Mechanics.* (*Fig.* 2.)

As if the unequal Radii *O M* and *O N* drawn from the centre *O* of any wheel, ſhould ſuſtain the weights *A* and *P*, by the cords *M A* and *N P*; and the forces of thoſe weights to move the wheel were required. Through the centre *O* draw the right line *K O L*, meeting the cords perpendicularly in *K* and *L*; and from the centre *O*, with *O L* the greater of the diſtances *O K* and *O L*, deſcribe a circle, meeting the cord *M A* in *D*: and drawing *O D*, make *A C* parallel and *D C* perpendicular thereto. Now, it being indifferent whether the points *K*, *L*, *D*, of the cords be fixed to the plane of the wheel or not, the weights will have the ſame effect whether they are ſuſpended from the points *K* and *L*, or from *D* and *L*. Let the whole force of the weight *A* be repreſented by the Line *A D*, and let it be reſolved into the forces *A C* and *C D*; of which the force *A C*, drawing the radius *O D* directly from the centre, will have no effect to move the wheel: but the other force *D C*, drawing the radius *D O* perpendicularly, will have the ſame effect as if it drew perpendicularly the radius *O L* equal to *O D*; that is, it will have the ſame effect as the weight *P*, if that weight is to the weight *A*, as the force *D C* is to the force *D A*; that is (becauſe of the ſimilar triangles *A D C*, *D O K*,) as *O K* to *O D* or *O L*. Therefore the weights *A* and *P*, which are reciprocally as the radii *O K* and *O L* that lye in the ſame right line, will be equipollent, and ſo remain in equilibrio: which is the well known property of the Ballance, the Lever, and the Wheel. If either weight is greater than in this ratio, its force to move the wheel will be ſo much the greater.

If the weight *p*, equal to the weight *P*, is partly ſuſpended by the cord *Np*, partly ſuſtained by the oblique plane *p G*; draw *p H*, *N H*, the former perpendicular to the horizon, the latter to the plane *p G*; and if the force of the weight *p* tending downwards is repreſented by the line *p H*, it may be reſolved into the forces *p N*, *H N*. If there was any plane perpendicular to the cord *p N*, cutting the other plane *p G* in a line parallel to the horizon; and the weight *p* was ſupported only by thoſe planes *p Q*, *p G*; it would preſs thoſe planes perpendicularly with the forces *p N*, *H N*; to wit, the plane *p Q* with the force *p N*, and the plane *p G* with the force *H N*. And therefore if the plane *p Q* was taken away, ſo that the weight might ſtretch the cord, becauſe the cord, now ſuſtaining the weight, ſupplies the place of the plane that was removed, it will be ſtrained by the ſame force *p N* which preſs'd upon the plane before. Therefore the tenſion of this oblique cord *p N* will be to that of the other perpendicular cord *P N* as *p N* to *p H*. And therefore if the weight *p* is to the weight *A* in a ratio compounded of the reciprocal ratio of the leaſt diſtances of the cords *p N*, *A M*, from the centre of the wheel,

and

and of the direct ratio of *pH* to *pN*; the weights will have the ſame effect towards moving the wheel, and will therefore ſuſtain each other, as any one may find by experiment.

But the weight *p* preſſing upon thoſe two oblique planes, may be conſider'd as a wedge between the two internal ſurfaces of a body ſplit by it; and hence the forces of the Wedge and the Mallet may be determin'd; for becauſe the force with which the weight *p* preſſes the plane *pQ*, is to the force with which the ſame, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line *p H* towards both the planes, as *pN* to *p H*; and to the force with which it preſſes the other plane *pG*, as *pN* to *N H*. And thus the force of the Screw may be deduced from a like reſolution of forces; it being no other than a Wedge impelled with the force of a Lever. Therefore the uſe of this Corollary ſpreads far and wide, and by that diffuſive extent the truth thereof is farther confirmed. For on what has been ſaid depends the whole doctrine of Mechanics variouſly demonſtrated by different authors. For from hence are eaſily deduced the forces of Machines, which are compounded of Wheels, Pulleys, Leavers, Cords and Weights, aſcending directly or obliquely, and other Mechanical Powers; as alſo the force of the Tendons to move the Bones of Animals.

COROLLARY III.

The Quantity of motion, which is collected by taking the ſum of the motions directed towards the ſame parts, and the difference of thoſe that are directed to contrary parts, ſuffers no change from the action of bodies among themſelves.

For Action and its oppoſite Re-action are equal, by Law 3, and therefore, by Law 2, they produce in the motions equal changes towards oppoſite parts. Therefore if the motions are directed towards the ſame parts, whatever is added to the motion of the preceding body will be ſubducted from the motion of that which follows; ſo that the ſum will be the ſame as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards oppoſite parts will remain the ſame.

Thus if a ſphærical body *A* with two parts of velocity is triple of a ſphærical body *B* which follows in the ſame right line with ten parts of velocity; the motion of *A* will be to that of *B*, as 6 to 10. Suppoſe then their motions to be of 6 parts and of 10 parts, and the ſum will be 16 parts. Therefore upon the meeting of the bodies, if *A* acquire 3, 4 or 5 parts of motion, *B* will loſe as many; and therefore after reflexion *A* will proceed with 9, 10 or 11 parts, and *B* with 7, 6 or 5 parts; the ſum remaining always of 16 parts as before. If the body *A* acquire 9, 10, 11 or 12 parts of motion, and therefore after meeting proceed with 15, 16, 17 or 18 parts; the body *B*, loſing

B, loſing ſo many parts as *A* has got, will either proceed with one part, having loſt 9; or ſtop and remain at reſt, as having loſt its whole progreſſive motion of 10 parts; or it will go back with one part, having not only loſt its whole motion, but (if I may ſo ſay) one part more; or it will go back with 2 parts, becauſe a progreſſive motion of 12 parts is took off. And ſo the Sums of the conſpiring motions 15+1, or 16+0, and the Differences of the contrary motions 17−1 and 18−2 will always be equal to 16 parts, as they were before the meeting and reflexion of the bodies. But, the motions being known with which the bodies proceed after reflexion, the velocity of either will be alſo known, by taking the velocity after to the velocity before reflexion, as the motion after is to the motion before. As in the laſt caſe, where the motion of the body *A* was of 6 parts before reflexion and of 18 parts after, and the velocity was of 2 parts before reflexion; the velocity thereof after reflexion will be found to be of 6 parts, by ſaying, as the 6 parts of motion before to 18 parts after, ſo are 2 parts of velocity before reflexion to 6 parts after.

But if the bodies are either not ſphærical, or moving in different right lines impinge obliquely one upon the other, and their motions after reflexion are required: in thoſe caſes we are firſt to determine the poſition of the plane that touches the concurring bodies in the point of concourſe; then the motion of each body (by Corol. 2.) is to be reſolved into two, one perpendicular to that plane, and the other parallel to it. This done, becauſe the bodies act upon each other in the direction of a line perpendicular to this plane, the parallel motions are to be retained the ſame after reflexion as before; and to the perpendicular motions we are to aſſign equal changes towards the contrary parts; in ſuch manner that the ſum of the conſpiring, and the difference of the contrary motions, may remain the ſame as before. From ſuch kind of reflexions alſo ſometimes ariſe the circular motions of bodies about their own centres. But theſe are caſes which I don't conſider in what follows; and it would be too tedious to demonſtrate every particular that relates to this ſubject.

COROLLARY IV.

The common centre of gravity of two or more bodies, does not alter its ſtate of motion or reſt by the actions of the bodies among themſelves; and therefore the common centre of gravity of all bodies acting upon each other (excluding outward actions and impediments) is either at reſt, or moves uniformly in a right line.

For if two points proceed with an uniform motion in right lines, and their diſtance be divided in a given ratio, the dividing point will be either at reſt, or proceed uniformly in a right line. This is demonſtrated hereafter in Lem. 23. and its Corol. when the points are moved in the ſame plane; and by a like way of arguing, it may be demonſtrated when the points are not moved in the ſame plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at reſt, or proceeds uniformly in a right line; becauſe the line which connects the centres of thoſe two bodies ſo moving is divided at that common centre in a given ratio. In like manner the common centre of thoſe two and that of a third body will be either at reſt or moving uniformly in a right line; becauſe at that centre, the diſtance between the common centre of the two bodies, and the centre of this laſt, is divided in a given ratio. In like manner the common centre of theſe three, and of a fourth body, is either at reſt, or moves uniformly in a right line; becauſe the diſtance between the common centre of the three bodies, and the centre of the fourth is there alſo divided in a given ratio, and ſo on *in infinitum*. Therefore in a ſyſtem of bodies, where there is neither any mutual action among themſelves, nor any foreign force impreſs'd upon them from without, and which conſequently move uniformly in right lines, the common centre of gravity of them all is either at reſt, or moves uniformly forwards in a right line.

Moreover, in a ſyſtem of two bodies mutually acting upon each other, ſince the diſtances between their centres and the common centre of gravity of both, are reciprocally as the bodies; the relative motions of thoſe bodies, whether of approaching to or of receding from that centre, will be equal among themſelves. Therefore ſince the changes which happen to motions are equal and directed to contrary parts, the common centre of thoſe bodies, by their mutual action between themſelves, is neither promoted nor retarded, nor ſuffers any change as to its ſtate of motion or reſt. But in a ſyſtem of ſeveral bodies, becauſe the common centre of gravity of any two acting mutually upon each other ſuffers no change in its ſtate by that action; and much leſs the common centre of gravity of the others with which that action does not intervene; but the diſtance between thoſe two centres is divided by the common centre of gravity of all the bodies into parts reciprocally proportional to the total ſums of thoſe bodies whoſe centres they are; and therefore while thoſe two centres retain their ſtate of motion or reſt, the common centre of all does alſo retain its ſtate: It is manifeſt, that the common centre of all never ſuffers any change in the ſtate of its motion or reſt from the actions of any two bodies between themſelves. But in ſuch a ſyſtem all the actions of the bodies among themſelves, either happen between two bodies, or are compoſed of actions interchanged between ſome two bodies; and therefore they do never produce any alteration in the common centre of all as to its ſtate of motion or reſt. Wherefore ſince that centre when the bodies do not act mutually one upon another, either is at reſt or moves uniformly forward in ſome right line; it will, notwithſtanding the mutual actions of the bodies among themſelves, always perſevere in its ſtate, either of reſt, or of proceeding uniformly in a right line, unleſs it is forc'd out of this ſtate by the action of ſome power impreſs'd from without upon the whole ſyſtem. And therefore the ſame law takes place in a ſyſtem, conſiſting of many bodies, as in one ſingle body, with regard to their perſevering in their ſtate of motion or of reſt. For the progreſſive motion whether of one ſingle body or of a whole ſyſtem of bodies, is always to be eſtimated, from the motion of the centre of gravity.

COROLLARY V.

The motions of bodies included in a given ſpace are the ſame among themſelves, whether that ſpace is at reſt, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the ſame parts, and the ſums of thoſe that tend towards contrary parts, are at firſt (by ſuppoſition) in both caſes the ſame; and it is from thoſe ſums and differences that the colliſions and impulſes do ariſe with which the bodies mutually impinge one upon another. Wherefore (by Law 2.) the effects of thoſe colliſions will be equal in both caſes; and therefore the mutual motions of the bodies among themſelves in the one caſe will remain equal to the mutual motions of the bodies among themſelves in the other. A clear proof of which we have from the experiment of a ſhip: where all motions happen after the ſame manner, whether the ſhip is at reſt, or is carried uniformly forwards in a right line.

COROLLARY VI.

If bodies, any how moved among themſelves are urged in the direction of parallel lines by equal accelerative forces; they will all continue to move among themſelves, after the ſame manner as if they had been urged by no ſuch forces.

For theſe forces acting equally (with reſpect to the quantities of the bodies to be moved) and in the direction of parallel lines, will (by Law 2.) move all the bodies equally (as to velocity) and therefore will never produce any change in the poſitions or motions of the bodies among themſelves.

SCHOLIUM.

Hitherto I have laid down ſuch principles as have been receiv'd by Mathematicians, and are confirm'd by abundance of experiments. By the two firſt Laws and the firſt two Corollaries, *Galileo* diſcover'd that the deſcent of bodies obſerv'd the duplicate ratio of the time, and that the motion of projectiles was in the curve of a Parabola; experience agreeing with both, unleſs ſo far as theſe motions are a little retarded by the reſiſtance of the Air. When a body is falling, the uniform force of its gravity acting equally, impreſſes, in equal particles of time, equal forces upon that body, and therefore generates equal velocities: and in the whole time impreſſes a whole force and generates a whole velocity, proportional to the time. And the ſpaces deſcribed in proportional times are as the velocities and the times conjunctly; that is, in a duplicate ratio of the times. And when a body is thrown upwards, its uniform gravity impreſſes forces and takes off velocities proportional to the times; and the times of aſcending to the greateſt heights are as the velocities to be taken off, and thoſe heights are as the velocities and the times conjunctly, or in the duplicate ratio of the velocities. And if a body be projected in any direction, the motion ariſing from its projection is compounded with the motion ariſing from its gravity. As if the Body *A* by its motion of projection alone (*Fig.* 3.) could deſcribe in a given time the right line *AB*, and with its motion of falling alone could deſcribe in the ſame time the altitude *AC*; compleat the paralellogram *ABDC*, and the body by that compounded motion will at the end of the time be found in the place *D*; and the curve line *AED*, which that body deſcribes, will be a Parabola, to which the right line *AB* will be a tangent in *A*; and whoſe ordinate *BD* will be as the ſquare of the line *AB*. On the ſame laws and corollaries depend thoſe things which have been demonſtrated concerning the times of the vibration of Pendulums, and are confirm'd by the daily experiments of Pendulum clocks. By the ſame together with the third Law Sir *Chriſt. Wren*, Dr. *Wallis* and Mr. *Huygens*, the greateſt Geometers of our times, did ſeverally determine the rules of the Congreſs and Reflexion of hard bodies, and much about the ſame time communicated their diſcoveries to the *Royal Society*, exactly agreeing among themſelves, as to thoſe rules. Dr. *Wallis* indeed was ſomething more early in the publication; then followed Sir *Chriſtopher Wren*, and laſtly, Mr. *Huygens*. But Sir *Chriſtopher Wren* confirmed the truth of the thing before the Royal Society by the experiment of pendulums, which Mr *Mariotte* ſoon after thought fit to explain in a treatiſe entirely upon that ſubject. But to bring this experiment to an accurate agreement with the theory, we are to have a due regard as well to the reſiſtance of the air, as to the elaſtic force of the concurring bodies. Let the ſphærical bodies *A B*, be ſuſpended by the parallel and equal ſtrings, *AC*, *B D*, *Fig.*4. from the centres *C*, *D*. About theſe centres, with thoſe intervals, deſcribe the ſemicircles *EAF*, *GBH* biſected by the radii *CA*, *D B*. Bring the body *A* to any point *R* of the arc *EAF*, and (withdrawing the body *B*) let it go from thence, and after one oſcillation ſuppoſe it to return to the point *V*: then *R V* will be the retardation ariſing from the reſiſtance of the air. Of this *R V* let *ST* be a fourth part ſituated in the middle, to wit, ſo as *R S* and *TV* may be equal, and *R S* may be to *ST* as 3 to 2 : then will *S T* repreſent very nearly the retardation during the deſcent from *S* to *A*. Reſtore the body *B* to its place : and ſuppoſing the body *A* to be let fall from the point *S*, the velocity thereof in the place of reflexion *A*, without ſenſible error, will be the ſame as if it had deſcended *in vacuo* from the point *T*. Upon which account this velocity may be repreſented by the chord of the arc *T A*. For it is a propoſition well known to Geometers, that the velocity of a pendulous body in the loweſt point is as the chord of the arc which it has deſcribed in its deſcent. After reflexion, ſuppoſe the body *A* comes to the place *s*, and the body *B* to the place *k*. Withdraw the body *B*, and find the place *v*, from which if the body *A*, being let go, ſhould after one oſcillation return to the place *r*, *s t* may be a fourth part of *r v*, ſo placed in the middle thereof as to leave *r s* equal to *t v*, and let the chord of the arc tA repreſent the velocity which the body A had in the place A immediately after reflexion. For t will be the true and correct place to which the Body A ſhould have aſcended, if the reſiſtance of the Air had been taken off. In the ſame way we are to correct the place k to which the body B aſcends, by finding the place l to which it ſhould have aſcended *in vacuo*. And thus every thing may be ſubjected to experiment, in the ſame manner as if we were really placed *in vacuo*. Theſe things being done we are to take the product (if I may ſo ſay) of the body A, by the chord of the arc TA (which repreſents its velocity) that we may have its motion in the place A immediately before reflexion; and then by the chord of the arc tA, that we may have its motion in the place A immediately after reflexion. And ſo we are to take the product of the body B by the chord of the arc Bl, that we may have the motion of the ſame immediately after reflexion. And in like manner, when two bodies are let go together from different places, we are to find the motion of each, as well before as after reflexion; and then we may compare the motions between themſelves, and collect the effects of the reflexion. Thus trying the thing with pendulums of ten feet, in unequal as well as equal bodies, and making the bodies to concur after a deſcent through large ſpaces, as of 8, 12, or 16 feet, I found always, without an errour of 3 inches, that when the bodies concurr'd together directly, equal changes toward the contrary parts were produced in their motions; and of conſequence, that the action and reaction were always equal. As if the body A imping'd upon the body B at reſt with 9 parts of motion, and loſing 7, proceeded after reflexion with 2; the body B was carried backwards with thoſe 7 parts. If the bodies concurr'd with contrary motions, *A* with twelve parts of motion, and *B* with ſix, then if *A* receded with 2, *B* receded with 8, to wit, with a deduction of 14 parts of motion on each ſide. For from the motion of *A* ſubducting 12 parts, nothing will remain: but ſubducting 2 parts more, a motion will be generated of 2 parts towards the contrary way; and ſo, from the motion of the body *B* of 6 parts, ſubducting 14 parts, a motion is generated of 8 parts towards the contrary way. But if the bodies were made both to move towards the ſame way; *A*, the ſwifter, with 14 parts of motion, *B*, the ſlower, with 5, and after reflexion *A* went on with 5, *B* likewiſe went on with 14 parts; 9 parts being transferr'd from *A* to *B*. And ſo in other caſes. By the congreſs and colliſion of bodies, the quantity of motion, collected from the ſum of the motions directed towards the ſame way, or from the difference of thoſe that were directed towards contrary ways, was never changed. For the error of an inch or two in meaſures may be eaſily aſcrib'd to the difficulty of executing every thing with accuracy. It was not eaſy to let go the two pendulums ſo exactly together, that the bodies ſhould impinge one upon the other in the lowermoſt place *AB*; nor to mark the places *s*, and *k*, to which the bodies aſcended after congreſs. Nay, and ſome errors too might have happen'd from the unequal denſity of the parts of the pendulous bodies themſelves, and from the irregularity of the texture proceeding from other cauſes.

But to prevent an objection that may perhaps be alledged againſt the rule, for the proof of which this experiment was made, as if this rule did ſuppoſe that the bodies were either abſolutely hard, or at leaſt perfectly elaſtic; whereas no ſuch bodies are to be found in nature; I muſt add that the experiments we have been deſcribing, by no means depending upon that quality of hardneſs, do ſucceed as well in ſoft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminiſh the reflexion in ſuch a certain proportion, as the quantity of the elaſtic force requires. By the theory of *Wren* and *Huygens*, bodies abſolutely hard return one from another with the ſame velocity with which they meet. But this may be affirm'd with more certainty of bodies perfectly elaſtic. In bodies imperfectly elaſtic the velocity of the return is to be diminiſh'd together with the elaſtic force; becauſe that force (except when the parts of bodies are bruiſed by their congreſs, or ſuffer ſome ſuch extenſion as happens under the ſtrokes of a hammer,) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly and ſtrongly compreſs'd. For firſt, by letting go the pendulous bodies and meaſuring their reflexion, I determined the quantity of their elaſtic force; and then, according to this force, eſtimated the reflexions that ought to happen in other caſes of congreſs. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity with which they met, as about 5 to 9. Balls of ſteel return'd with almoſt the ſame velocity: thoſe of cork with a velocity ſomething leſs: but in balls of glaſs the proportion was as about 15 to 16. And thus the third law, ſo far as it regards percuſſions and reflexions, is prov'd by a theory, exactly agreeing with experience.

In attractions, I briefly demonſtrate the thing after this manner. Suppoſe an obſtacle is interpos'd to hinder the congreſs of any two bodies *A*,*B*, mutually attracting one the other: then if either body as *A*, is more attracted towards the other body *B*, than that other body *B* is towards the firſt body *A*, the obſtacle will be more ſtrongly urged by the preſſure of the body *A* than by the preſſure of the body *B*; and therefore will not remain in æquilibrio: but the ſtronger preſſure will prevail, and will make the ſyſtem of the two bodies, together with the obſtacle, to move directly towards the parts on which *B* lies; and in free ſpaces, to go forward ***in infinitum*** with a motion perpetually accelerated. Which is abſurd, and contrary to the firſt law. For by the firſt law, the ſyſtem ought to perſevere in it's ſtate of reſt, or of moving uniformly forward in a right line; and therefore the bodies muſt equally preſs the obſtacle, and be equally attracted one by the other. I made the experiment on the loadſtone and iron. If theſe plac'd apart in proper veſſels, are made to float by one another in ſtanding water; neither of them will propell the other, but by being equally attracted, they will ſuſtain each others preſſure, and reſt at laſt in an equilibrium.

So the gravitation betwixt the Earth and its parts, is mutual. Let the Earth *FI* (*Fig.* 5.) be cut by any plane *EG* into two parts *EGF* and *EGI*: and their weights one towards the other will be mutually equal. For if by another plane *HK*, parallel to the former *EG*, the greater part *EGI* is cut into two parts *EGKH* and *HKI*, whereof *HKI* is equal to the part *EFG* firſt cut off: it is evident that the middle part *EGKH* will have no propenſion by its proper weight towards either ſide, but will hang as it were and reſt in an equilibrium betwixt both. But the one extreme part *HKI* will with its whole weight bear upon and preſs the middle part toward the other extreme part EGF; and therefore the force, with which EGI, the ſum of the parts HKI and $EGKH$, tends towards the third part EGF, is equal to the weight of the part HKI, that is, to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF, one towards the other, are equal, as I was to prove. And indeed if thoſe weights were not equal, the whole Earth floating in the non-reſiſting æther, would give way to the greater weight, and retiring from it, wou'd be carried off *in infinitum*.

And as thoſe bodies are equipollent in the congreſs and reflexion, whoſe velocities are reciprocally as their innate forces: ſo in the uſe of mechanic inſtruments, thoſe agents are equipollent and mutually ſuſtain each the contrary preſſure of the other, whoſe velocities, eſtimated according to the determination of the forces, are reciprocally as the forces.

So thoſe weights are of equal force to move the arms of a Ballance, which during the play of the ballance are reciprocally as their velocities upwards and downwards: that is, if the aſcent or deſcent is direct, thoſe weights are of equal force, which are reciprocally as the diſtances of the points at which they are ſuſpended from the axe of the ballance; but if they are turned aſide by the interpoſition of oblique planes, or other obſtacles, and made to aſcend or deſcend obliquely, thoſe bodies will be equipollent, which are reciprocally as the heights of their aſcent and deſcent taken according to the perpendicular; and that on account of the determination of gravity downwards.

And in like manner in the Pully, or in a combination of Pullies, the force of a hand drawing the rope directly, that is to the weight, whether aſcending directly or obliquely, as the velocity of the perpendicular aſcent of the weight to the velocity of the hand that draws the rope, will ſuſtain the weight.

In Clocks and ſuch like inſtruments, made up from a combination of wheels, the contrary forces that promote and impede the motion of the wheels, if they are reciprocally as the velocities of the parts of the wheel on which they are impreſs'd, will mutually ſuſtain the one the other.

The force of the Screw to preſs a body is to the force of the hand that turns the handles by which it is moved, as the circular velocity of the handle in that part where it is impelled by the hand, is to the progreſſive velocity of the Screw towards the preſs'd body.

The forces by which the Wedge preſſes or drives the two parts of the wood it cleaves, are to the force of the mallet upon the wedge, as the progreſs of the wedge in the direction of the force impreſs'd upon it by the mallet, is to the velocity with which the parts of the wood yield to the wedge, in the direction of lines perpendicular to the ſides of the wedge. And the like account is to be given of all Machines.

The power and uſe of Machines conſiſts only in this, that by diminiſhing the velocity we may augment the force, and the contrary: From whence in all ſorts of proper Machines, we have the ſolution of this problem; *To move a given weight with a given power*, or with a given force to overcome any other given reſiſtance. For if Machines are ſo contriv'd, that the velocities of the agent and reſiſtant are reciprocally as their forces; the agent will juſt ſuſtain the reſiſtant: but with a greater diſparity of velocity will overcome it. So that if the diſparity of velocities is ſo great, as to overcome all that reſiſtance, which commonly ariſes either from the attrition of contiguous bodies as they ſlide by one another, or from the coheſion of continuous bodies that are to be ſeparated, or from the weights of bodies to be raiſed; the exceſs of the force remaining, after all thoſe reſiſtances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the Machine, as in the reſiſting body. But to treat of Mechanics is not my preſent buſineſs. I was only willing to ſhew by thoſe examples, the great extent and certainty of the third law of motion. For if we eſtimate the action of the agent from its force and velocity conjunctly; and likewiſe the re-action of the impediment conjunctly from the velocities of its ſeveral parts, and from the forces of reſiſtance ariſing from the attrition, coheſion, weight, and acceleration of thoſe parts; the action and re-action in the uſe of all ſorts of Machines will be found always equal to one another. And ſo far as the action is propagated by the intervening inſtruments, and at laſt impreſs'd upon the reſiſting body, the ultimate determination of the action will be always contrary to the determination of the re-action.

OF THE MOTION OF BODIES.

BOOK I.

SECTION I.

Of the method of firſt and laſt ratio's of quantities, by the help whereof we demonſtrate the propoſitions that follow.

LEMMA I.

Quantities, and the ratio's of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other

other than by any given difference, become ultimately equal.

If you deny it; ſuppoſe them to be ultimately unequal, and let *D* be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference *D*; which is againſt the ſuppoſition.

LEMMA II.

If in any figure AacE (Pl. 1. Fig. 6.) *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inſcrib'd any number of parallelograms* Ab, Bc, Cd, *&c. comprehended under equal baſes* AB, BC, CD, *&c. and the ſides* Bb, Cc, Dd, *&c. parallel to one ſide* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c. are compleated. Then if the breadth of thoſe parallelograms be ſuppos'd to be diminiſhed, and their number to be augmented* in infinitum: *I ſay that the ultimate ratio's which the inſcrib'd figure* AKbLcMdD, *the circumſcribed figure* AalbmcndoE, *and curvilinear figure* AabcdE, *will have to one another, are ratio's of equality.*

For the difference of the inſcrib'd and circumſcrib'd figures is the ſum of the parallelograms *Kl, Lm, Mn, Do*, that is, (from the equality of all their baſes) the rectangle under one of their baſes *Kb* and the ſum of their altitudes *Aa*, that is, the rectangle *ABla*.

But this rectangle, becauſe its breadth *AB* is ſuppos'd diminiſhed *in infinitum*, becomes leſs than any given ſpace. And therefore (by Lem. 1.) the figures inſcribed and circumſcribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. *Q. E. D.*

LEMMA III.

The ſame ultimate ratio's are alſo ratio's of equality, when the breadths AB, BC, DC, *&c. of the parallelograms are unequal, and are all diminiſhed* in infinitum.

For ſuppoſe *AF* equal to the greateſt breadth, and compleat the parallelogram *FAaf.* This parallelogram will be greater than the difference of the inſcrib'd and circumſcribed figures; but, becauſe its breadth *AF* is diminiſhed *in infinitum*, it will become leſs than any given rectangle. *Q. E. D.*

COR. 1. Hence the ultimate ſum of thoſe evaneſcent parallelograms will in all parts coincide with the curvilinear figure.

COR. 2. Much more will the rectilinear figure, comprehended under the chords of the evaneſcent arcs *ab*, *bc*, *cd*, &c. ultimately coincide with the curvilinear figure.

COR. 3. And alſo the circumſcrib'd rectilinear figure comprehended under the tangents of the ſame arcs.

COR. 4. And therefore theſe ultimate figures (as to their perimeters *acE*,) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures Aa c E, Ppr T, (Pl. 1. Fig. 7.) *you inſcribe (as before) two ranks of parallelograms, an equal number in each rank, and when their breadths are diminiſhed* in infinitum, *the ultimate ratio's of the parallelograms in one figure to thoſe in the other, each to each reſpectively, are the ſame; I ſay that thoſe two figures* AacE, PprT, *are to one another in that ſame ratio.*

For as the parallelograms in the one are ſeverally to the parallelograms in the other, ſo (by compoſition) is the ſum of all in the one to the ſum of all in the other; and ſo is the one figure to the other, becauſe (by Lem. 3.) the former figure to the former ſum, and the latter figure to the latter ſum are both in the ratio of equality. *Q. E. D.*

COR. Hence if two quantities of any kind are any how divided into an equal number of parts: and thoſe parts, when their number is augmented and their magnitude diminiſhed *in infinitum,* have a given ratio one to the other, the firſt to the firſt, the ſecond to the ſecond, and ſo on in order: the whole quantities will be one to the other in that ſame given ratio. For if, in the figures of this lemma, the parallelograms are taken one to the other in the ratio of the parts, the ſum of the parts will always be as the ſum of the parallelograms; and therefore ſuppoſing the number

Plate I. Vol. I. P. 44.

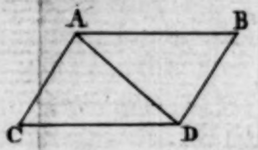


Fig. 1. p. 21.

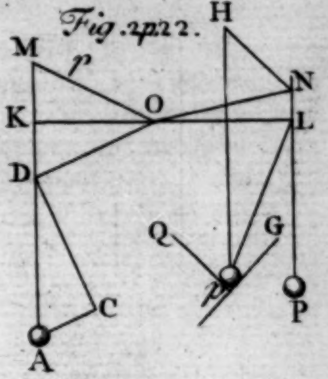


Fig. 2. p. 22.

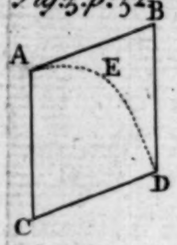


Fig. 3. p. 32.

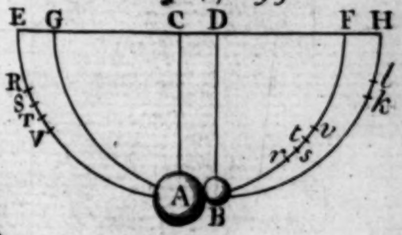


Fig. 4. p. 33.

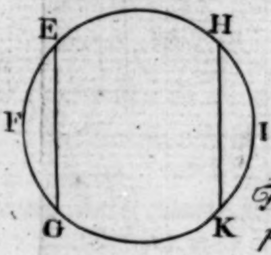


Fig. 5. p. 37.

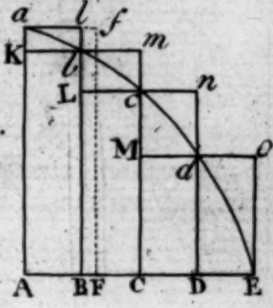


Fig. 6. p. 42.

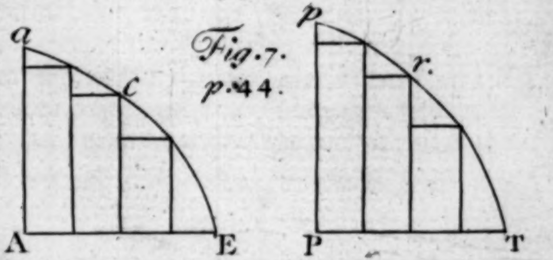


Fig. 7. p. 44.

number of the parallelograms and parts to be augmented, and their magnitudes diminiſhed *in infinitum*, thoſe ſums will be in the ultimate ratio of the parallelogram in the one figure to the correſpondent parallelogram in the other; that is, (by the ſuppoſition) in the ultimate ratio of any part of the one quantity to the correſpondent part of the other.

LEMMA V.

In ſimilar figures, all ſorts of homologous ſides, whether curvilinear or rectilinear, are proportional; and the area's are in the duplicate ratio of the homologous ſides.

LEMMA VI.

If any arc ACB (Pl.2.Fig.1.) *given in poſition is ſubtended by its chord* AB, *and in any point* A *in the middle of the continued curvature, is touch'd by a right line* AD, *produced both ways; then if the points* A *and* B *approach one another and meet, I ſay the angle* BAD, *contained between the chord and the tangent, will be diminiſhed* in infinitum, *and ultimately will vaniſh.*

For if that angle does not vaniſh, the arc *ACB* will contain with the tangent *AD* an angle equal to a rectilinear angle; and therefore the curvature at the point *A* will not be continued, which is againſt the ſuppoſition.

LEM.

LEMMA VII.

The ſame things being ſuppoſed; I ſay, that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality. Pl. 2. Fig. 1.

For while the point *B* approaches towards the point *A*, conſider always *AB* and *AD* as produc'd to the remote points *b* and *d*, and parallel to the ſecant *BD* draw *bd*: and let the arc *Acb* be always ſimilar to the arc *ACB*. Then ſuppoſing the points *A* and *B* to coincide, the angle *dAb* will vaniſh, by the preceding lemma; and therefore the right lines *Ab*, *Ad* (which are always finite) and the intermediate arc *Acb* will coincide, and become equal among themſelves. Whereſore the right lines *AB*, *AD*, and the intermediate arc *ACB* (which are always proportional to the former) will vaniſh; and ultimately acquire the ratio of equality. *Q. E. D.*

COR. 1. Whence if through *B* (*Pl.* 2. *Fig.* 2.) we draw *BF* parallel to the tangent, always cutting any right line *AF* paſſing through *A* in *F*; this line *BF* will be ultimately in the ratio of equality with the evaneſcent arc *ACB*; becauſe, compleating the parallelogram *AFBD*, it is always in a ratio of equality with *AD*.

COR. 2. And if through *B* and *A* more right lines are drawn as *BE*, *BD*, *AF*, *AG* cutting the tangent *AD* and its parallel *BF*; the ultimate ratio of all the abſciſſa's *AD*, *AE*, *BF*, *BG*, and of the chord and arc *AB*, any one to any other, will be the ratio of equality.

Cor. 3. And therefore in all our reaſoning about ultimate ratio's, we may freely uſe any one of thoſe lines for any other.

Lemma VIII.

If the right lines AR, BR, (Pl. 2. Fig. 1.) *with the arc* ACB, *the chord* AB *and the tangent* AD, *conſtitute three triangles* RAB, RACB, RAD, *and the points* A *and* B *approach and meet: I ſay that the ultimate form of theſe evaneſcent triangles is that of ſimilitude, and their ultimate ratio that of equality.*

For while the point *B* approaches towards the point *A* conſider always *AB*, *AD*, *AR*, as produced to the remote points *b*, *d*, and *r*, and *r b d* as drawn parallel to *R D*, and let the arc *A c b* be always ſimilar to the arc *A C B*. Then ſuppoſing the points *A* and *B* to coincide, the angle *b Ad* will vaniſh; and therefore the three triangles *r A b*, *r A c b*, *r Ad*, (which are always finite) will coincide, and on that account become both ſimilar and equal. And therefore the triangles *R AB*, *R ACB*, *R AD*, which are always ſimilar and proportional to theſe, will ultimately become both ſimilar and equal among themſelves. *Q. E. D.*

Cor. And hence in all our reaſonings about ultimate ratio's, we may indifferently uſe any one of thoſe triangles for any other.

LEMMA IX.

If a right line AE, (Pl. 2. Fig. 3.) *and a curve line* ABC, *both given by poſition, cut each other in a given angle* A; *and to that right line, in another given angle,* BD, CE *are ordinately applied, meeting the curve in* B, C; *and the points* B *and* C *together, approach towards, and meet in the point* A: *I ſay that the area's of the triangles* ABD, ACE, *will ultimately be one to the other in the duplicate ratio of the ſides.*

For while the points *B*, *C* approach towards the point *A*, ſuppoſe always *AD* to be produced to the remote points *d* and *e*, ſo as *Ad*, *Ae* may be proportional to *AD*, *AE*; and the ordinates *db*, *ec*, to be drawn parallel to the ordinates *DB* and *EC*, and meeting *AB* and *AC* produced in *b* and *c*. Let the curve *Abc* be ſimilar to the curve *ABC*, and draw the right line *Ag* ſo as to touch both curves in *A*, and cut the ordinates *DB*, *EC*, *db*, *ec*, in *F*, *G*, *f*, *g*. Then, ſuppoſing the length *Ae* to remain the ſame, let the points *B* and *C* meet in the point *A*; and the angle *cAg* vaniſhing, the curvilinear areas *Abd*, *Ace* will coincide with the rectilinear areas *Afd*, *Age*; and therefore (by Lem. 5) will be one to the other in the duplicate ratio of the ſides *Ad*, *Ae*. But the areas *ABD*, *ACE* are always proportional to theſe areas; and ſo the ſides *AD*, *AE* are to theſe ſides. And therefore the areas *ABD*, *ACE* are ultimately one to the other in the duplicate ratio of the ſides *AD*, *AE*. *Q.E.D.*

LEMMA X.

The ſpaces which a body deſcribes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminiſhed, are in the very beginning of the motion one to the other in the duplicate ratio of the times.

Let the times be repreſented by the lines *AD*, *AE*, and the velocities generated in thoſe times by the ordinates *DB*, *EC*. The ſpaces deſcribed with theſe velocities will be as the areas *ABD*, *ACE*, deſcribed by thoſe ordinates, that is, at the very beginning of the motion (by Lem. 9.) in the duplicate ratio of the times *AD*, *AE*. *Q.E.D.*

COR. 1. And hence one may eaſily infer, that the errors of bodies deſcribing ſimilar parts of ſimilar figures in proportional times, are nearly in the duplicate ratio of the times in which they are generated; if ſo be theſe errors are generated by any equal forces ſimilarly applied to the bodies, and meaſur'd by the diſtances of the bodies from thoſe places of the ſimilar figures, at which, without the action of thoſe forces, the bodies would have arrived in thoſe proportional times.

Cor. 2. But the errors that are generated by proportional forces ſimilarly applied to the bodies at ſimilar parts of the ſimilar figures, are as the forces and the ſquares of the times conjunctly.

Cor. 3. The ſame thing is to be underſtood of any ſpaces whatſoever deſcrib'd by bodies urged with different forces. All which, in the very beginning of the motion, are as the forces and the ſquares of the times conjunctly.

Cor. 4. And therefore the forces are as the ſpaces deſcribed in the very beginning of the motion directly and the ſquares of the times inverſly.

Cor. 5. And the ſquares of the times are as the ſpaces deſcrib'd directly and the forces inverſly.

SCHOLIUM.

If in comparing indetermined quantities of different ſorts one with another, any one is ſaid to be as any other directly or inverſly: the meaning is, that the former is augmented or diminiſhed in the ſame ratio with the latter, or with its reciprocal. And if any one is ſaid to be as any other two or more directly or inverſly: the meaning is, that the firſt is augmented or diminiſhed in the ratio compounded of the ratio's in which the others, or the reciprocals of the others, are augmented or diminiſhed. As if A is ſaid to be as B directly and C directly and D inverſly: the meaning is, that A is augmented or diminiſhed in the ſame ratio with $B \times C \times \frac{1}{D}$, that is to ſay, that A and $\frac{BC}{D}$ are one to the other in a given ratio.

Lem.

LEMMA XI.

The evaneſcent ſubtenſe of the angle of contact, in all curves, which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the ſubtenſe of the conterminate arc. Pl. 2. Fig. 4.

CASE 1. Let AB be that arc, AD its tangent, BD the ſubtenſe of the angle of contact perpendicular on the tangent, AB the ſubtenſe of the arc. Draw BG perpendicular to the ſubtenſe AB, and AG to the tangent AD, meeting in G; then let the points D, B, and G, approach to the points d, b, and g, and ſuppoſe J to be the ultimate interſection of the lines BG, AG, when the points D, B have come to A. It is evident that the diſtance GJ may be leſs than any aſſignable. But (from the nature of the circles paſſing through the points A, B, G; A, b, g) $AB^2 = AG \times BD$, and $Ab^2 = Ag \times bd$; and therefore the ratio of AB^2 to Ab^2 is compounded of the ratio's of AG to Ag and of BD to bd. But becauſe GJ may be aſſum'd of leſs length than any aſſignable, the ratio of AG to Ag may be ſuch as to differ from the ratio of equality by leſs than any aſſignable difference; and therefore the ratio of AB^2 to Ab^2 may be ſuch as to differ from the ratio of BD to bd by leſs than any aſſignable difference. Therefore, by Lem. 1. the ultimate ratio of AB^2 to Ab^2 is the ſame with the ultimate ratio of BD to bd. Q. E. D.

CASE 2. Now let BD be inclined to AD in any given angle, and the ultimate ratio of BD to bd will always be the ſame as before, and therefore the ſame with the ratio of AB^2 to Ab^2. *Q.E.D.*

CASE 3. And if we ſuppoſe the angle *D* not to be given, but that the right line *BD* converges to a given point, or is determined by any other condition whatever; nevertheleſs, the angles *D*, *d*, being determined by the ſame law, will always draw nearer to equality, and approach nearer to each other than by any aſſigned difference, and therefore, by Lem. 1, will at laſt be equal, and therefore the lines *BD*, *bd* are in the ſame ratio to each other as before. *Q. E. D.*

COR. 1. Therefore ſince the tangents *AD*, *Ad*, the arcs *AB*, *Ab*, and their ſines *BC*, *bc*, become ultimately equal to the chords *AB*, *Ab*; their ſquares will ultimately become as the ſubtenſes *BD*, *bd*.

COR. 2. Their ſquares are alſo ultimately as the verſed ſines of the arcs, biſecting the chords, and converging to a given point. For thoſe verſed ſines are as the ſubtenſes *BD*, *bd*.

COR. 3. And therefore the verſed ſine is in the duplicate ratio of the time in which a body will deſcribe the arc with a given velocity.

COR. 4. The rectilinear triangles *ADB*, *Adb* are ultimately in the triplicate ratio of the ſides *AD*, *Ad*, and in a ſeſquiplicate ratio of the ſides *DB*, *db*; as being in the ratio compounded of the ſides *AD* to *DB*, and of *Ad* to *db*. So alſo the triangles *ABC*, *Abc* are ultimately in the triplicate ratio of the ſides *BC*, *bc*. What I call the ſeſquiplicate ratio is the ſubduplicate of the triplicate, as being compounded of the ſimple and ſubduplicate ratio.

COR. 5. And becauſe *DB*, *db* are ultimately parallel and in the duplicate ratio of the lines *AD*, *Ad*: the ultimate curvilinear areas *ADB*, *Adb* will be (by the nature of the parabola) two thirds of the rectilinear triangles *ADB*, *Adb*; and the ſegments *AB*, *Ab* will be one third of the ſame triangles. And thence thoſe areas and thoſe ſegments will be in the triplicate ratio as well of the tangents *AD*, *Ad*; as of the chords and arcs *AB*, *Ab*.

SCHOLIUM.

But we have all along ſuppoſed the angle of contact to be neither infinitely greater nor infinitely leſs, than the angles of contact made by circles and their tangents; that is, that the curvature at the point *A* is neither infinitely ſmall nor infinitely great, or that the interval *AJ* is of a finite magnitude. For *DB* may be taken as AD^3: in which caſe no circle can be drawn through the point *A*, between the tangent *AD* and the curve *AB*, and therefore the angle of contact will be infinitely leſs than thoſe of circles. And by a like reaſoning if *DB* be made ſucceſſively as AD, AD^5, AD^6, AD, *&c.* we ſhall have a ſeries of angles of contact, proceeding *in infinitum*, wherein every ſucceeding term is infinitely leſs than the preceding. And if *DB* be made ſucceſſively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, *&c.* we ſhall have another infinite ſeries of angles of contact, the firſt of which is of the ſame ſort with thoſe of circles, the ſecond infinitely greater, and every ſucceeding one infinitely greater than the preceding. But between any two of theſe angles another ſeries of intermediate angles of contact may be interpoſed proceeding both ways *in infinitum*, wherein every ſucceeding angle ſhall be infinitely greater, or infinitely leſs than the preceding. As if between the terms AD^2 and AD^3 there were interpoſed the ſeries $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again between any two angles of this ſeries, a new ſeries of intermediate angles may be interpoſed, differing from one another by infinite intervals. Nor is nature confin'd to any bounds.

Thoſe things which have been demonſtrated of curve lines and the ſuperficies which they comprehend, may be eaſily applied to the curve ſuperficies and contents of ſolids. Theſe lemmas are premiſed, to avoid the tedious neſs of deducing perplexed demonſtrations *ad abſurdum*, according to the method of the ancient geometers. For demonſtrations are more contracted by the method of indiviſibles: But becauſe the hypotheſis of indiviſibles ſeems ſomewhat harſh, and therefore that method is reckoned leſs geometrical; I choſe rather to reduce the demonſtrations of the following propoſitions to the firſt and laſt ſums and ratio's of naſcent and evaneſcent quantities, that is, to the limits of thoſe ſums and ratio's; and ſo to premiſe, as ſhort as I could, the demonſtrations of thoſe limits. For hereby the ſame thing is perform'd as by the method of indiviſibles; and now thoſe principles being demonſtrated, we may uſe them with more ſafety. Therefore if hereafter, I ſhould happen to conſider quantities as made up of particles, or ſhould uſe little curve lines for right ones; I would not be underſtood to mean indiviſibles, but evaneſcent diviſible quantities; not the ſums and ratio's of determinate parts, but always the limits of ſums and ratio's: and that the force of ſuch demonſtrations always depends on the method lay'd down in the foregoing lemma's.

Perhaps it may be objected, that there is no ultimate proportion of evaneſcent quantities; becauſe the proportion, before the quantities have vaniſhed, is not the ultimate, and when they are vaniſhed, is none. But by the ſame argument it may be alledged, that a body arriving at a certain place, and there ſtopping, has no ultimate velocity: becauſe the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the anſwer is eaſy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its laſt place and the motion ceaſes, nor after, but at the very inſtant it arrives; that is, that velocity with which the body arrives at its laſt place, and with which the motion ceaſes. And in like manner, by the ultimate ratio of evaneſcent quantities is to be underſtood the ratio of the quantities, not before they vaniſh, nor afterwards, but with which they vaniſh. In like manner the firſt ratio of naſcent quantities is that with which they begin to be. And the firſt or laſt ſum is that with which they begin and ceaſe to be (or to be augmented or diminiſhed.) There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and ceaſe to be. And ſince ſuch limits are certain and definite, to determine the ſame is a problem ſtrictly geometrical. But whatever is geometrical we may be allowed to uſe in determining and demonſtrating any other thing that is likewiſe geometrical.

It may alſo be objected, that if the ultimate ratio's of evaneſcent quantities are given, their ultimate magnitudes will be alſo given: and ſo all quantities will conſiſt of indiviſibles, which is contrary to what *Euclid* has demonſtrated concerning incommenſurables, in the 10th book of his Elements. But this objection is founded on a falſe ſuppoſition. For thoſe ultimate ratio's with which quantities vaniſh, are not truly the ratio's of ultimate quantities, but limits towards which the ratio's of quantities, decreaſing without limit, do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminiſhed *in infinitum.* This thing will appear more evident in quantities infinitely great. If two quantities, whoſe difference is given, be augmented *in infinitum*, the ultimate ratio of theſe quantities will be given, to wit, the ratio of equality; but it does not from thence follow, that the ultimate or greateſt quantities themſelves, whoſe ratio that is, will be given. Therefore if in what follows, for the ſake of being more eaſily underſtood, I ſhould happen to mention quantities as leaſt, or evaneſcent, or ultimate; you are not to ſuppoſe that quantities of any determinate magnitude are meant, but ſuch as are conceiv'd to be always diminiſhed without end.

SECTION II.

Of the invention of centripetal forces.

Proposition I. Theorem I.

The areas, which revolving bodies deſcribe by radii drawn to an immoveable centre of force, do lie in the ſame immoveable planes, and are proportional to the times in which they are deſcribed. Pl. 2. Fig. 5.

FOR ſuppoſe the time to be divided into equal parts, and in the firſt part of that time, let the body by its innate force deſcribe the right line *AB* In the ſecond part of that time, the ſame would, (by law 1.) if not hinder'd, proceed directly to *c*, along the line *Bc* equal to *AB*; ſo that by the radii *AS*, *BS*, *cS* drawn to the centre, the equal areas *ASB*, *BSc*, would be deſcribed. But when the body is arrived at *B*, ſuppoſe that a centripetal force acts at once with a great impulſe, and turning aſide the body from the right line *B c*, compells it afterwards to continue its motion along the right line *BC*. Draw *c C* parallel to *BS* meeting *BC* in *C*; and at the end of the ſecond part of the time, the body (by Cor. 1. of the laws) will be found in *C*, in the ſame plane with the triangle *ASB*. Joyn *SC*, and, becauſe *SB* and *Cc* are parallel, the triangle *SBC* will be equal to the triangle *SBc*, and therefore alſo to the triangle *SAB*. By the like argument, if the centripetal force acts ſucceſſively in *C, D, E,* &c. and makes the body in each ſingle particle of time, to deſcribe the right lines *CD, DE, EF,* &c. they will all lye in the ſame plane; and the triangle *SCD* will be equal to the triangle *SBC*, and *SDE* to *SCD*, and *SEF* to *SDE*. And therefore in equal times, equal areas are deſcrib'd in one immoveable plane: and, by compoſition, any ſums *SADS*, *SAFS*, of thoſe areas, are one to the other, as the times in which they are deſcrib'd. Now let the number of thoſe triangles be augmented, and their breadth diminiſhed *in infinitum*; and (by cor. 4. lem. 3.) their ultimate perimeter *ADF* will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any deſcrib'd areas *SADS*, *SAFS*, which are always proportional to the times of deſcription, will, in this caſe alſo, be proportional to thoſe times. *Q.E.D.*

Cor. 1. The velocity of a body attracted towards an immoveable centre, in ſpaces void of reſiſtance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in thoſe places *A, B, C, D, E* are as the baſes *AB, BC, CD, DE, EF*, of equal triangles; and theſe baſes are reciprocally as the perpendiculars let fall upon them.

Cor. 2. If the chords *AB, BC* of two arcs, ſucceſſively deſcribed in equal times, by the ſame body, in ſpaces void of reſiſtance, are compleated into a parallelogram *ABCV*, and the diagonal *BV* of this parallelogram, in the poſition which it ultimately acquires when thoſe arcs are diminiſhed *in infinitum*, is produced both ways, it will paſs through the centre of force.

COR. 3. If the chords *AB*, *BC*, and *DE*, *EF*, of arcs deſcrib'd in equal times, in ſpaces void of reſiſtance, are compleated into the parallelograms *ABCV*, *DEFZ*; the forces in *B* and *E* are one to the other in the ultimate ratio of the diagonals *BV*, *EZ*, when thoſe arcs are diminiſhed *in infinitum.* For the motions *BC* and *EF* of the body (by cor. 1. of the laws) are compounded of the motions *Bc*, *BV*, and *Ef*, *EZ*: but *BV* and *EZ*, which are equal to *Cc* and *Ff*, in the demonſtration of this propoſition, were generated by the impulſes of the centripetal force in *B* and *E*, and are therefore proportional to thoſe impulſes.

COR. 4. The forces by which bodies, in ſpaces void of reſiſtance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are one to another as the vers'd ſines of arcs deſcribed in equal times; which verſed ſines tend to the centre of force, and biſect the chords when thoſe arcs are diminiſhed to infinity. For ſuch vers'd ſines are the halfs of the diagonals mentioned in cor. 3.

COR. 5. And therefore thoſe forces are to the force of gravity, as the ſaid vers'd ſines to the vers'd ſines perpendicular to the horizon of thoſe parabolic arcs which projectiles deſcribe in the ſame time.

COR. 6. And the ſame things do all hold good (by cor. 5. of the laws) when the planes in which the bodies are mov'd, together with the centres of force which are placed in thoſe planes, are not at reſt but move uniformly forward in right lines.

PROP.

PROPOSITION II. THEOREM II.

Every body, that moves in any curve line deſcribed in a plane, and by a radius, drawn to a point either immoveable, or moving forward with an uniform rectilinear motion, deſcribes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

CASE 1. For every body that moves in a curve line, is (by law 1.) turned aſide from its rectilinear courſe by the action of ſome force that impels it. And that force by which the body is turned off from its rectilinear courſe, and is made to deſcribe, in equal times, the equal leaſt triangles *SAB*, *SBC*, *SCD*, *&c.* about the immoveable point *S*, (by prop. 40. book 1. elem. and law 2.) acts in the place *B*, according to the direction of a line parallel to *cC*, that is, in the direction of the line *BS*; and in the place *C*, according to the direction of a line parallel to *dD*, that is, in the direction of the line *CS*, &c. And therefore acts always in the direction of lines tending to the immoveable point *S*. *Q.E.D.*

CASE 2. And (by cor. 5. of the laws) it is indifferent whether the ſuperficies in which a body deſcribes a curvilinear figure be quieſcent, or moves together with the body, the figure deſcrib'd, and its point *S*, uniformly forwards in right lines.

COR.

Fig. 1. p. 45. 47.

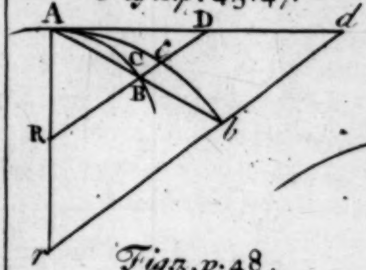


Fig. 2. p. 46.

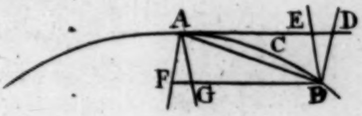


Fig. 3. p. 48.

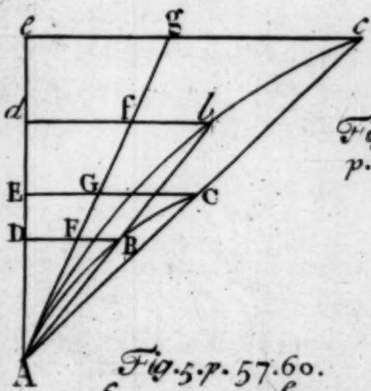


Fig. 4. p. 51.

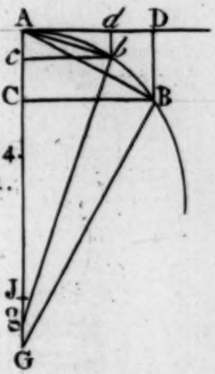
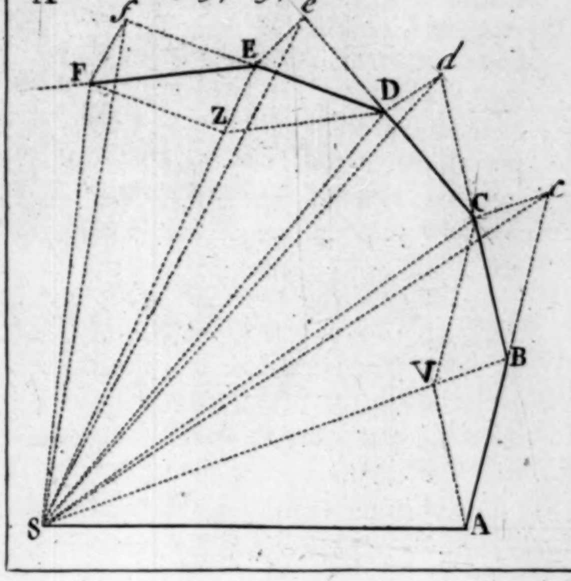


Fig. 5. p. 57. 60.



COR. 1. In non-reſiſting ſpaces or mediums, if the areas are not proportional to the times, the forces are not directed to the point in which the radii meet; but deviate therefrom *in conſequentia*, or towards the parts to which the motion is directed, if the deſcription of the areas is accelerated; but *in antecedentia*, if retarded.

COR. 2. And even in reſiſting mediums, if the deſcription of the areas is accelerated, the directions of the forces deviate from the point in which the radii meet, towards the parts to which the motion tends.

SCHOLIUM.

A body may be urged by a centripetal force compounded of ſeveral forces. In which caſe the meaning of the propoſition is, that the force which reſults out of all, tends to the point *S*. But if any force, acts perpetually in the direction of lines perpendicular to the deſcrib'd ſurface; this force will make the body to deviate from the plane of its motion: but will neither augment nor diminiſh the quantity of the deſcribed ſurface, and is therefore to be neglected in the compoſition of forces.

PROPOSITION III. THEOREM III.

Every body, that, by a radius drawn to the centre of another body howſoever moved, deſcribes areas about that centre proportional to the times, is urged by a force compounded out of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

Let *L* repreſent the one, and *T* the other body; and (by Cor. 6 of the laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the ſecond body *T* is urgeed, the firſt body *L* will go on to deſcribe about the other body *T*, the ſame areas as before: but the force, by which that other body *T* was urged, will be now deſtroyed by an equal and contrary force; and therefore (by Law 1.) that other body *T*, now left to it ſelf, will either reſt, or move uniformly forward in a right line: and the firſt body *L* impell'd by the difference of the forces, that is, by the force remaining, will go on to deſcribe about the other body *T*, areas proportional to the times. And therefore (by Theor. 2.) the difference of the forces is directed to the other body *T*, as its centre. *Q.E.D.*

COR. 1. Hence if the one body *L*, by a radius drawn to the other body *T*, deſcribes areas proportional to the times; and from the whole force, by which the firſt body *L* is urged (whether that force is ſimple, or, according to cor. 2. of the laws, compounded out of ſeveral forces) we ſubduct (by the ſame cor.) that whole accelerative force, by which the other body is urged; the whole remaining force by which the firſt body is urged, will tend to the other body *T*, as its centre.

Cor. 2. And, if theſe areas are proportional to the times nearly, the remaining force will tend to the other body *T* nearly.

Cor. 3. And *vice verſa*, if the remaining force tends nearly to the other body *T*, thoſe areas will be nearly proportional to the times.

Cor. 4. If the body *L*, by a radius drawn to the other body *T*, deſcribes areas, which compared with the times, are very unequal; and that other body *T* be either at reſt or moves uniformly forward in a right line: the action of the centripetal force tending to that other body *T*, is either none at all, or it is mix'd and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immoveable or moveable) centre. The ſame thing obtains, when the other body is moved by any motion whatſoever; provided that centripetal force is taken, which remains after ſubducting that whole force acting upon that other body *T*.

Scholium.

Becauſe the equable deſcription of areas indicates that a centre is reſpected by that force with which the body is moſt affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit:

why may we not be allowed in the following diſcourſe, to uſe the equable deſcription of areas as an indication of a centre, about which all circular motion is performed in free ſpaces?

PROPOSITION IV. THEOREM IV.

The centripetal forces of bodies, which by equable motions deſcribe different circles, tend to the centres of the ſame circles; and are one to the other, as the ſquares of the arcs deſcribed in equal times applied to the radii of the circles.

Theſe forces tend to the centres of the circles (by prop. 2. and cor. 2. prop. 1) and are one to another as the verſed ſines of the leaſt arcs deſcribed in equal times (by cor. 4. prop. 1.) that is, as the ſquares of the ſame arcs applied to the diameters of the circles, (by lem. 7.) and therefore ſince thoſe arcs are as arcs deſcribed in any equal times, and the diameters are as the radii; the forces will be as the ſquares of any arcs deſcribed in the ſame time applied to the radii of the circles. *Q.E.D.*

COR. 1. Therefore, ſince thoſe arcs are as the velocities of the bodies, the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the ſimple ratio of the radii inverſely.

COR. 2. And, ſince the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inverſely; the centripetal forces are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodic times inverſely.

COR. 3. Whence if the periodic times are equal, and the velocities therefore as the radii; the centripetal forces will be alſo as the radii; and the contrary.

COR. 4. If the periodic times and the velocities are both in the ſubduplicate ratio of the radii; the centripetal forces will be equal among themſelves: and the contrary.

COR. 5. If the periodic times are as the radii, and therefore the velocities equal; the centripetal forces will be reciprocally as the radii: and the contrary.

COR. 6. If the periodic times are in the ſeſquiplicate ratio of the radii, and therefore the velocities reciprocally in the ſubduplicate ratio of the radii; the centripetal forces will be in the duplicate ratio of the radii inverſely: and the contrary.

COR. 7. And univerſally, if the periodic time is as any power R^{n} of the radius *R*, and therefore the velocity reciprocally as the power R^{n-1} of the radius; the centripetal force will be reciprocally as the power R^{2n-1} of the radius: and the contrary.

COR. 8. The ſame things all hold concerning the times, the velocities, and forces by which bodies deſcribe the ſimilar parts of any ſimilar figures, that have their centres in a ſimilar poſition within thoſe figures; as appears by applying the demonſtration of the preceding caſes to thoſe. And the application is eaſy by only ſubſtituting the equable deſcription of areas in the place of equable motion, and uſing the diſtances of the bodies from the centres inſtead of the radii.

Cor. 9. From the ſame demonſtration it likewiſe follows, that the arc which a body, uniformly revolving in a circle by means of a given centripetal force, deſcribes in any time, is a mean proportional between the diameter of the circle, and the ſpace which the ſame body falling by the ſame given force would deſcend thro' in the ſame given time.

SCHOLIUM.

The caſe of the 6th corollary obtains in the celeſtial bodies, (as Sir *Chriſtopher Wren*, Dr. *Hooke*, and Dr. *Halley* have ſeverally obſerved) and therefore in what follows, I intend to treat more at large of thoſe things which relate to centripetal force decreaſing in a duplicate ratio of the diſtances from the centres.

Moreover, by means of the preceding propoſition and its corollaries, we may diſcover the proportion of a centripetal force to any other known force, ſuch as that of gravity. For if a body by means of its gravity revolves in a circle concentric to the Earth, this gravity is the centripetal force of that body. But from the deſcent of heavy bodies, the time of one entire revolution, as well as the arc deſcribed in any given time, is given, (by cor. 9. of this prop.) And by ſuch propoſitions, Mr. *Huygens*, in his excellent book *De Horologio Oſcillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding propoſition may be likewiſe demonſtrated after this manner. In any circle ſuppoſe a polygon to be inſcribed of any number of ſides. And if a body, moved with a given velocity along the ſides of the polygon, is reflected from the circle at the ſeveral angular points; the force, with which at every reflection it ſtrikes the circle, will be as its velocity: and therefore the ſum of the forces, in a given time, will be as that velocity and the number of reflexions conjunctly; that is, (if the ſpecies of the polygon be given) as the length deſcribed in that given time, and increaſed or diminiſhed in the ratio of the ſame length to the radius of the circle; that is, as the ſquare of that length applied to the radius: and therefore if the polygon, by having its ſides diminiſhed *in infinitum*, coincides with the circle, as the ſquare of the arc deſcribed in a given time applied to the radius. This is the centrifugal force, with which the body impells the circle; and to which the contrary force, wherewith the circle continually repells the body towards the centre, is equal.

PROPOSITION V. PROBLEM I.

There being given in any places, the velocity with which a body deſcribes a given figure, by means of forces directed to ſome common centre; to find that centre. Pl. 3. Fig. 1.

Let the three right lines *PT*, *TQV*, *VR* touch the figure deſcribed in as many points *P*, *Q*, *R*, and meet in *T* and *V*. On the tangents erect the perpendiculars *PA*, *QB*, *RC*, reciprocally proportional to the velocities of the body in the points *P*, *Q*, *R*, from which the perpendiculars were raiſed; that is, ſo that *PA* may be to *QB* as the velocity in *Q* to the velocity in *P*, and *QB* to *RC* as the velocity in *R* to the velocity in *Q*. Thro' the ends *A*, *B*, *C*, of the perpendiculars draw *AD*, *DBE*, *EC*, at right angles, meeting in *D* and *E*: And the right lines *TD*, *VE* produced, will meet in *S* the centre required.

For the perpendiculars let ſall from the centre *S* on the tangents *PT*, *QT*, are reciprocally as the velocities of the bodies in the points *P* and *Q* (by cor. 1. prop. 1.) and therefore, by conſtruction, as the perpendiculars *AP*, *BQ* directly; that is, as the perpendiculars let fall from the point *D* on the tangents. Whence it is eaſy to infer, that the points *S*, *D*, *T*, are in one right line. And by the like argument the points *S*, *E*, *V* are alſo in one right line; and therefore the centre *S* is in the point where the right lines *TD*, *VE* meet. *Q. E. D.*

PROPOSITION VI. THEOREM V.

In a ſpace void of reſiſtance, if a body revolves in any orbit about an immoveable centre, and in the leaſt time deſcribes any arc juſt then naſcent; and the verſed ſine of that arc is ſuppoſed to be drawn, biſecting the chord, and produced paſſing through the centre of force: the centripetal force in the middle of the arc, will be as the verſed ſine directly and the ſquare of the time inverſely.

For the verſed ſine in a given time is as the force (by cor. 4. prop. 1.) and augmenting the time in any ratio, becauſe the arc will be augmented in the ſame ratio, the verſed ſine will be augmented in the duplicate of that ratio, (by cor. 2 and 3. lem. 11.) and therefore is as the force and the ſquare of the time. Subduct on both ſides the duplicate ratio of the time, and the force will be as the verſed ſine directly and the ſquare of the time inverſely. *Q. E. D.*

And the ſame thing may alſo be eaſily demonſtrated by corol. 4. lem. 10.

COR. 1. If a body *P* revolving about the centre *S*, (*Pl.* 3. *Fig.* 2.) deſcribes a curve line *APQ*, which a right line *ZPR* touches in any point *P*; and from any other point *Q* of the curve, *QR* is drawn parallel to the diſtance *SP*, meeting the tangent in *R*; and *QT* is drawn perpendicular to the diſtance *SP*: the centripetal force will be reciprocally as the ſolid $\frac{SP^2 \times QT^2}{QR}$, if the ſolid be taken of that magnitude which it ultimately acquires when the points *P* and *Q* coincide. For *QR* is equal to the verſed ſine of double the arc *QP*, whoſe middle is *P*: and double the triangle *SQP*, or $SP \times QT$ is proportional to the time, in which that double arc is deſcribed; and therefore may be uſed for the exponent of the time.

COR. 2. By a like reaſoning, the centripetal force is reciprocally as the ſolid $\frac{SY^2 \times QP^2}{QR}$; if *SY* is a perpendicular from the centre of force on *PR* the tangent of the orbit. For the rectangles $SY \times QP$ and $SP \times QT$ are equal.

COR. 3. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle the leaſt angle of contact or ſection, having the ſame curvature and the ſame radius of curvature at the point *P*; and if *PV* be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be reciprocally as the ſolid $SY^2 \times PV$. For *PV* is $\frac{QP^2}{QR}$.

Cor. 4. The ſame things being ſuppoſed, the centripetal force is as the ſquare of the velocity directly, and that chord inverſely. For the velocity is reciprocally as the perpendicular *SY*, by cor. 1, prop. 1.

Cor. 5. Hence if any curvilinear figure *APQ* is given; and therein a point *S* is alſo given to which a centripetal force is perpetually directed; that law of centripetal force may be found, by which the body *P* will be continually drawn back from a rectilinear courſe, and being detained in the perimeter of that figure, will deſcribe the ſame by a perpetual revolution. That is, we are to find by computation, either the ſolid $\frac{SP^2 \times QT^2}{QR}$ or the ſolid $SY^2 \times PV$, reciprocally proportional to this force. Examples of this we ſhall give in the following problems.

Proposition VII. Problem II.

If a body revolves in the circumference of a circle; it is propoſed to find the law of centripetal force directed to any given point. Pl. 3. Fig. 3.

Let *VQPA* be the circumference of the circle; *S* the given point to which as to a centre the force tends; *P* the body moving in the circumference; *Q* the next place into which it is to move; and *PRZ* the tangent of the circle at the preceding place. Through the point *S* draw the chord *PV*, and the diameter *VA* of the circle, join *AP*, and draw *QT* perpendicular to *SP*, which produced, may meet the tangent PR in Z; and laſtly, thro' the point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, becauſe of the ſimilar triangles ZQR, ZTP, VPA, we ſhall have RP^2, that is, QRL, to QT^2, as AV^2 to PV^2. And therefore $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2. Multiply thoſe equals by $\frac{SP^2}{QR}$, and the points P and Q coinciding, for RL write PV; then we ſhall have $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. And therefore (by cor. 1. and 5. prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$, that is, (becauſe AV^2 is given) reciprocally as the ſquare of the diſtance or altitude SP, and the cube of the chord PV conjunctly. *Q. E. I.*

The ſame otherwiſe.

On the tangent PR produced, let fall the perpendicular SY: and (becauſe of the ſimilar triangles SYP, VPA) we ſhall have AV to PV as SP to SY, and therefore $\frac{SP \times PV}{AV} = SY$, and $\frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV$. And therefore (by corol. 3 and 5. prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is, (becauſe AV is given) reciprocally as $SP^2 \times PV^3$. *Q. E. I.*

Cor. 1. Hence if the given point *S*, to which the centripetal force always tends, is placed in the circumference of the circle, as at *V*; the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude *SP*.

Cor. 2. The force by which the body *P* in the circle *APTV* (*Pl.* 3. *Fig.* 4.) revolves about the centre of force *S* is to the force by which the ſame body *P* may revolve in the ſame circle and in the ſame periodic time about any other centre of force *R*, as $RP^2 \times SP$ to the cube of the right line *SG*, which from the firſt centre of force *S*, is drawn parallel to the diſtance *PR* of the body from the ſecond centre of force *R*, meeting the tangent *PG* of the orbit in *G*. For by the conſtruction of this propoſition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$; that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$ or, (becauſe of the ſimilar triangles *PSG*, *TPV*) to SG^3.

Cor. 3. The force by which the body *P* in any orbit revolves about the centre of force *S*, is to the force by which the ſame body may revolve in the ſame orbit, and in the ſame periodic time about any other centre of force *R*, as the ſolid $SP \times RP^2$, contained under the diſtance of the body from the firſt centre of force *S*, and the ſquare of its diſtance from the ſecond centre of force *R*, to the cube of the right line *SG*, drawn from the firſt centre of force *S*, parallel to the diſtance *RP* of the body from the ſecond centre of force *R*, meeting the tangent *PG* of the orbit in *G*. For the force in this orbit at any point *P* is the ſame as in a circle of the ſame curvature.

PROPOSITION VIII. PROBLEM III.

If a body moves in the ſemi-circumference PQA; *it is propoſed to find the law of the centripetal force tending to a point* S, *ſo remote, that all the lines* PS, RS *drawn thereto, may be taken for parallels.* Pl. 3. Fig. 5.

From C the centre of the ſemi-circle, let the ſemidiameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Becauſe of the ſimilar triangles CPM, PZT and RZQ we ſhall have CP^2 to PM^2 as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or the points P, Q coinciding, to the rectangle $QR \times 2PM$. Therefore CP^2 is to PM^2 as $QR \times 2PM$ to QT^2; and $\frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$. And therefore (by corol. 1. and 5. prop. 6.) the centripetal force is reciprocally as $\frac{2PM^3 \times SP^2}{CP^2}$; that is, (neglecting the given ratio $\frac{2SP^2}{CP^2}$) reciprocally as PM^3. *Q. E. I.*

And the ſame thing is likewiſe eaſily inferred from the preceding Propoſition.

SCHOLIUM.

And by a like reaſoning, a body will be moved in an ellipſis, or even in an hyperbola, or parabola, by a centripetal force which is reciprocally as the cube of the ordinate directed to an infinitely remote centre of force.

PROPOSITION IX. PROBLEM IV.

If a body revolves in a ſpiral PQS, *cutting all the radii* SP, SQ, *&c. in a given angle: it is propoſed to find the law of the centripetal force tending to the centre of that ſpiral.* Pl. 3. Fig. 6.

Suppoſe the indefinitely ſmall angle PSQ to be given; becauſe then all the angles are given, the figure $SPRQT$ will be given in ſpecie. Therefore the ratio $\frac{QT}{QR}$ is alſo given, and $\frac{QT^2}{QR}$ is as QT, that is (becauſe the figure is given in ſpecie) as SP. But if the angle PSQ is any way changed, the right line QR, ſubtending the angle of contact QPR, (by lem. 11.) will be changed in the duplicate ratio of PR or QT. Therefore the ratio $\frac{QT^2}{QR}$ remains the ſame as before, that is as SP. And $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and therefore (by corol. 1. and 5. prop. 6.) the centripetal force is reciprocally as the cube of the diſtance SP. *Q. E. I.*

The

The ſame otherwiſe.

The perpendicular ST let fall upon the tangent, and the chord PV of the circle concentrically cutting the ſpiral are in given ratio's to the height SP; and therefore SP^3 is as $ST^2 \times PV$, that is (by corol. 3. and 5. prop. 6.) reciprocally as the centripetal force.

LEMMA XII.

All parallelograms circumſcribed about any conjugate diameters of a given ellipſis or hyperbola are equal among themſelves.

This is demonſtrated by the writers on the conic ſections.

PROPOSITION X. PROBLEM V.

If a body revolves in an ellipſis: it is propoſed to find the law of the centripetal force tending to the centre of the ellipſis. Pl. 4. Fig. 1.

Suppoſe CA, CB to be ſemi-axes of the ellipſis; GP, DK conjugate diameters; PF, QT perpendiculars to thoſe diameters; Qv an ordinate to the diameter GP; and if the parallelogram $QvPR$ be compleated; then (by the properties of the conic ſections) the rectangle PvG will be to Qv^2 as PC^2 to CD^2, and (becauſe of the ſimilar triangles QvT, PCF) Qv^2 to QT^2 as PC^2 to PF^2; and by compoſition, the ratio of PvG to QT^2 is compounded of the ratio of PC^2 to CD^2 and of the ratio of PC^2 to PF^2, that is, vG to $\frac{QT^2}{Pv}$ as PC^2 to $\frac{CD^2 \times PF^2}{PC^2}$. Put QR for Pv, and (by lem. 12.) $BC \times CA$ for $CD \times PF$, alſo (the points P and Q coinciding,) $2PC$ for vG; and multiplying the extremes and means together, we ſhall have $\frac{QT^2 \times PC^2}{QR}$ equal to $\frac{2BC^2 \times CA^2}{PC}$. Therefore (by cor. 5. prop. 6.) the centripetal force is reciprocally as $\frac{2BC^2 \times CA^2}{PC}$; that is (becauſe $2BC^2 \times CA^2$ is given) reciprocally as $\frac{1}{PC}$; that is, directly as the diſtance PC. *Q.E.I.*

The ſame otherwiſe.

In the right line PG on the other ſide of the point T, take the point u ſo that Tu may be equal to Tv; then take uV, ſuch as ſhall be to vG as DC^2 to PC^2. And becauſe Qv^2 is to PvG as DC^2 to PC^2, (by the conic ſections) we ſhall have $Qv^2 = Pv \times uV$. Add the rectangle uPv to both ſides, and the ſquare of the chord of the arc PQ will be equal to the rectangle VPv; and thereſore a circle, which touches the conic ſection in P, and paſſes thro' the point Q, will paſs alſo thro' the point V. Now let the points P and Q meet, and the ratio of uV to vG, which is the ſame with the ratio of DC^2 to PC^2, will become the ratio of PV to PG or PV to $2PC$; and therefore PV will be equal to $\frac{2DC^2}{PC}$. And therefore the force, by which the body P revolves in the ellipſis, will be reciprocally as

Plate III. Vol. I. P. 76.

Fig. 1. p. 67.

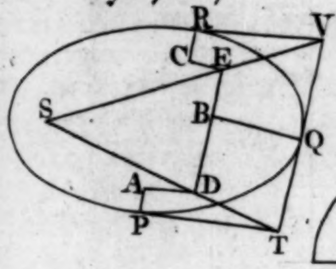


Fig. 2. p. 69.

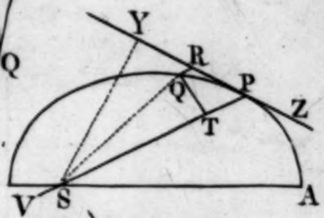


Fig. 3. p. 70.

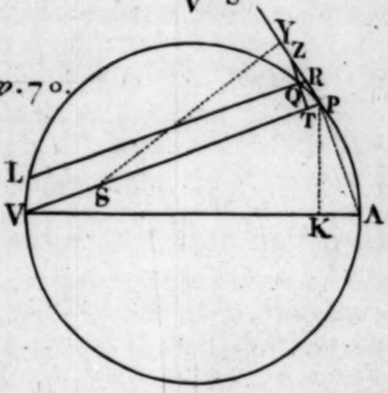


Fig. 4. p. 72.

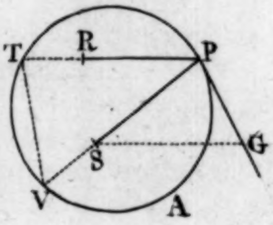


Fig. 5. p. 73.

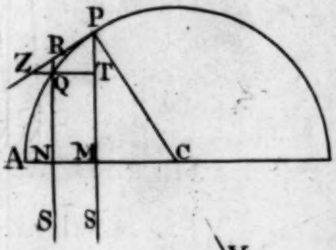
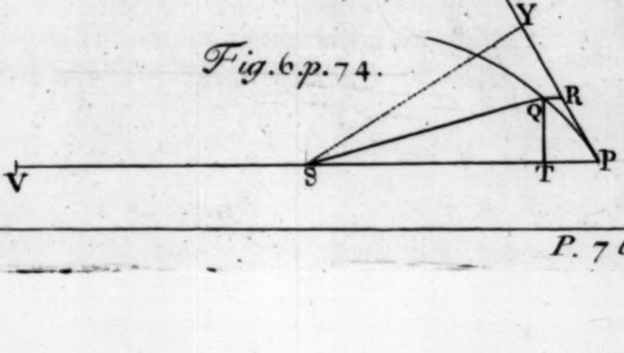


Fig. 6. p. 74.



$\frac{2DC^2}{PC} \times PF^2$ (by cor. 3. prop. 6.) that is, (becauſe $2DC^2 \times PF^2$ is given) directly as PC. *Q. E. I.*

COR. 1. And therefore the force is as the diſtance of the body from the centre of the ellipſis; and *vice verſa* if the force is as the diſtance, the body will move in an ellipſis whoſe centre coincides with the centre of force, or perhaps in a circle into which the ellipſis may degenerate.

COR. 2. And the periodic times of the revolutions made in all ellipſes whatſoever about the ſame centre will be equal. For thoſe times in ſimilar ellipſes will be equal (by corol. 3 and 8. prop. 4.) but in ellipſes that have their greater axe common, they are one to another as the whole areas of the ellipſes directly, and the parts of the areas deſcribed in the ſame time inverſely; that is, as the leſſer axes directly, and the velocities of the bodies in their principal vertices inverſely; that is, as thoſe leſſer axes directly, and the ordinates to the ſamepoint of the common axis inverſely; and therefore (becauſe of the equality of the direct and inverſe ratio's) in the ratio of equality.

SCHOLIUM.

If the ellipſis by having its centre removed to an infinite diſtance degenerates into a parabola, the body will move in this parabola; and the force, now tending to a centre infinitely remote, will become equable. Which is *Galileo*'s theorem. And if the parabolic ſection of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipſis, if the forces are directed to the centre of the figure placed in the abſciſſa, thoſe forces by increaſing or diminiſhing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abſciſſa, are always augmented or diminiſhed in the ratio of the diſtances from the centre; provided the periodic times remain equal; ſo alſo in all figures whatſoever, if the ordinates are augmented or diminiſhed in any given ratio, or their inclination is any way changed, the periodic time remaining the ſame; the forces directed to any centre placed in the abſciſſa, are in the ſeveral ordinates augmented or diminiſhed in the ratio of the diſtances from the centre.

SECTION III.

Of the motion of bodies in eccentric Conic ſections.

PROPOSITION XI. PROBLEM VI.

If a body revolves in an ellipſis: it is required to find the law of the centripetal force tending to the focus of the ellipſis. Pl. 4. Fig. 2.

Let S be the focus of the ellipſis. Draw SP cutting the diameter DK of the ellipſis in E, and the ordinate Qv in x; and compleat the parallelogram $QxPR$. It is evident that EP is equal to the greater ſemi-axis AC: for drawing HI from the other focus H of the ellipſis parallel to EC, becauſe CS, CH are equal ES, EI will be alſo equal, ſo that EP is the half ſum of PS, PI, that is, (becauſe of the parallels HI, PR, and the equal angles IPR, HPZ) of PS, PH, which taken together are equal to the whole axis $2AC$. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipſis (or for $\frac{2BC^2}{AC}$,) we ſhall have $L \times QR$ to

$L \times Pv$

$L \times Pv$ as QR to Pv, that is, as PE or AC to PC; and $L \times Pv$ to GvP as L to Gv; and GvP to Qv^2 as PC^2 to CD^2; and (by corol. 2. lem. 7.) the points Q and P coinciding, Qv^2 is to Qx^2 in the ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2 or (by lem. 12.) as CD^2 to CB^2. And compounding all thoſe ratio's together, we ſhall have $L \times QR$ to QT^2 as $AC \times L \times PC^2 \times CD^2$ or $2\,CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as $2\,PC$ to Gv. But the points Q and P coinciding, $2\,PC$ and Gv are equal. And therefore the quantities $L \times QR$ and QT^2, proportional to theſe, will be alſo equal. Let thoſe equals be drawn into $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by corol. 1. and 5. prop. 6.) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ratio of the diſtance SP. *Q. E. I.*

The ſame otherwiſe.

Seeing the force tending to the centre of the ellipſis, by which the body P may revolve in that ellipſis, is (by corol. 1. prop. 10.) as the diſtance CP of the body from the centre C of the ellipſis; let CE be drawn parallel to the tangent PR of the ellipſis; and the force, by which the ſame body P may revolve about any other point S of the ellipſis, if CE and PS interſect in E, will be as $\frac{PE^3}{SP^2}$ (by cor. 3. prop. 7.) that is, if the point S is the focus of the ellipſis, and therefore PE be given, as SP^2 reciprocally. *Q. E. I.*

Plate IV. *Vol*. I. *P*. 80.

Fig. 1. *p*. 75.

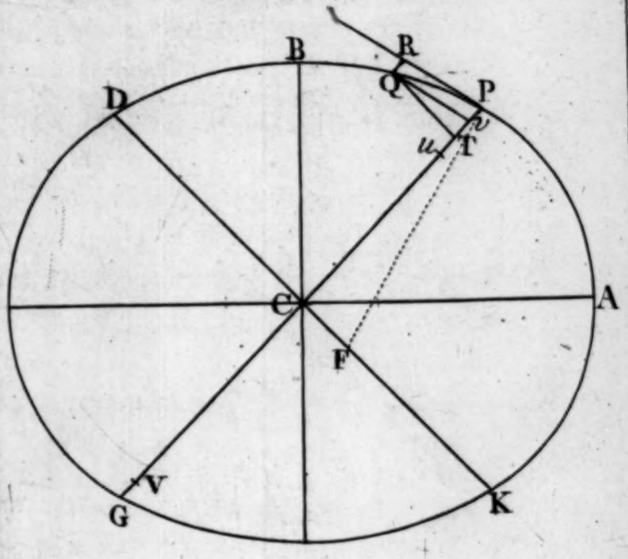
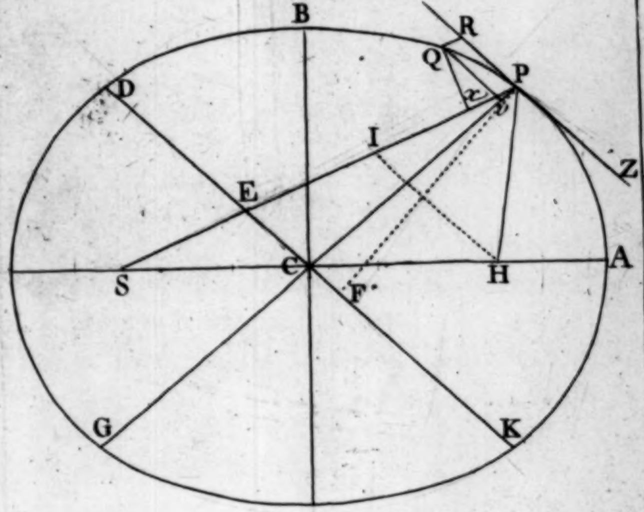


Fig. 2. *p*. 79.



With the ſame brevity with which we reduced the fifth problem to the parabola and hyperbola, we might do the like here: But becauſe of the dignity of the problem and its uſe in what follows, I ſhall confirm the other caſes by particular demonſtrations.

PROPOSITION XII. PROBLEM VII.

Suppoſe a body to move in an hyperbola: it is required to find the law of the centripetal force tending to the focus of that figure. Pl. 5. Fig. 1.

Let CA, CB be the ſemi-axes of the hyperbola; PG, KD other conjugate diameters; PF a perpendicular to the diameter KD; and Qv an ordinate to the diameter GP. Draw SP cutting the diameter DK in E, and the ordinate Qv in x, and compleat the parallelogram $QRPx$. It is evident that EP is equal to the ſemi-tranſverſe axe AC; for, drawing HI, from the other focus H of the hyperbola, parallel to EC, becauſe CS, CH are equal, ES, EI will be alſo equal; ſo that EP is the half difference of PS, PI; that is, (becauſe of the parallels IH, PR, and the equal angles IPR, HPZ) of PS, PH, the difference of which is equal to the whole axis $2AC$. Draw QT perpendicular to SP. And putting L for the principal latus rectum of the hyperbola, (that is, for $\frac{2BC^2}{AC}$,) we ſhall have $L \times QR$ to $L \times Pv$ as QR to Pv, or Px to Pv, that is, (becauſe of the ſimilar triangles Pxv, PEC) as PE to PC, or AC to PC. And

$L \times Pv$ will be to $Gv \times Pv$ as L to Gv; and (by the properties of the conic ſections) the rectangle GvP is to Qv^2 as PC^2 to CD^2; and (by cor. 2. lem. 7.) Qv^2 to Qx^2, the points Q and P coinciding, becomes a ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by lem. 12.) as CD^2 to CB^2: and, compounding all thoſe ratio's together, we ſhall have $L \times QR$ to QT^2 as $AC \times L \times PC^2 \times CD^2$ or $2CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as $2PC$ to Gv. But the points P and Q coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \times QR$ and QT^2, proportional to them, will be alſo equal. Let thoſe equals be drawn into $\frac{SP^2}{QR}$, and we ſhall have $L \times SP^2$ equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by cor. 1 & 5. prop. 6.) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ratio of the diſtance SP. *Q. E. I.*

The ſame otherwiſe.

Find out the force tending from the centre C of the hyperbola. This will be proportional to the diſtance CP. But from thence (by cor. 3. prop. 7.) the force tending to the focus S will be as $\frac{PE^3}{SP^2}$, that is, becauſe PE is given, reciprocally as SP^2. *Q. E. I.*

And the ſame way it may be demonſtrated, that the body having its centripetal changed into a centrifugal force, will move in the conjugate hyperbola.

LEMMA

LEMMA XIII.

The latus rectum of a parabola belonging to any vertex is quadruple the diſtance of that vertex from the focus of the figure.

This is demonſtrated by the writers on the conic ſections.

LEMMA XIV.

The perpendicular let fall from the focus of a parabola on its tangent, is a mean proportional between the diſtances of the focus from the point of contact, and from the principal vertex of the figure. Pl. 5. Fig. 2.

For, let AP be the parabola, S its focus, A its principal vertex, P the point of contact, PO an ordinate to the principal diameter, PM the tangent meeting the principal diameter in M, and SN the perpendicular from the focus on the tangent. Join AN, and becauſe of the equal lines MS and SP, MN and NP, MA and AO; the right lines AN, OP, will be parallel; and thence the triangle SAN will be right angled at A, and ſimilar to the equal triangles SNM, SNP: therefore PS is to SN as SN to SA. *Q. E. D.*

COR. 1. PS^2 is to SN^2 as PS to SA.

COR. 2. And becauſe SA is given, SN^2 will be as PS.

COR. 3. And the concourſe of any tangent PM with the right line SN, drawn from the focus perpendicular on the tangent, falls in the right line AN, that touches the parabola in the principal vertex.

PROPOSITION XIII. PROBLEM VIII.

If a body moves in the perimeter of a parabola: it is required to find the law of the centripetal force tending to the focus of that figure. Pl. 5. Fig. 3.

Retaining the conſtruction of the preceding lemma, let P be the body in the perimeter of the parabola; and from the place Q, into which it is next to ſucceed, draw QR parallel and QT perpendicular to SP, as alſo Qv parallel to the tangent, and meeting the diameter YG in v, and the diſtance SP in x. Now, becauſe of the ſimilar triangles Pxv, SPM, and of the equal ſides SP, SM of the one, the ſides Px or QR and Pv of the other will be alſo equal. But (by the conic ſections) the ſquare of the ordinate Qv is equal to the rectangle under the latus rectum and the ſegment Pv of the diameter, that is, (by lem. 13.) to the rectangle $4\,PS \times Pv$, or $4\,PS \times QR$; and the points P and Q coinciding, the ratio of Qv to Qx (by cor. 2. lem. 7.) becomes a ratio of equality. And therefore Qx^2, in this caſe, becomes equal to the rectangle $4\,PS \times QR$. But (becauſe of the ſimilar triangles QxT, SPN) Qx^2 is to QT^2 as PS^2 to SN^2, that is by cor. 1. lem. 14.) as PS to SA; that is, as $4\,PS \times QR$ to $4\,SA \times QR$, and therefore (by prop. 9. lib. 5. elem.) QT^2 and $4SA \times QR$ are equal. Multiply theſe equals by $\frac{SP^2}{QR}$, and $\frac{SP^2 \times QT^2}{QR}$ will become equal to $SP^2 \times 4SA$: and therefore (by cor. 1. and 5. prop. 6.) the centripetal force is reciprocally as $SP^2 \times 4SA$; that is, becauſe $4SA$ is given, reciprocally in the duplicate ratio of the diſtance *SP*. *Q. E. I.*

COR. 1. From the three laſt propoſitions it follows, that if any body *P* goes from the place *P* with any velocity in the direction of any right line *P R*, and at the ſame time is urged by the action of a centripetal force, that is reciprocally proportional to the ſquare of the diſtance of the places from the centre; the body will move in one of the conic ſections, having its focus in the centre of force; and the contrary. For the focus, the point of contact, and the poſition of the tangent being given, a conic ſection may be deſcribed, which at that point ſhall have a given curvature. But the curvature is given from the centripetal force and the bodies velocity given: and two orbits mutually touching one the other, cannot be deſcribed by the ſame centripetal force and the ſame velocity.

COR. 2. If the velocity, with which the body goes from its place *P*, is ſuch, that in any infinitely ſmall moment of time the lineola *PR* may be thereby deſcribed; and the centripetal force ſuch as in the ſame time to move that body through the ſpace *QR*; the body will move in one of the conic ſections, whoſe principal latus rectum is the quantity $\frac{QT^2}{QR}$ in its ultimate ſtate, when the lineolæ *P R*, *Q R* are diminiſhed *in infinitum*. In theſe corollaries, I conſider the circle as an ellipſis; and I except the caſe, where the body deſcends to the centre in a right line.

PROPOSITION XIV. THEOREM VI.

If ſeveral bodies revolve about one common centre, and the centripetal force is reciprocally in the duplicate ratio of the diſtance of places from the centre; I ſay, that the principal latera recta of their orbits are in the duplicate ratio of the area's, which the bodies by radii drawn to the centre deſcribe in the ſame time. Pl. 6. Fig. 1.

For (by cor. 2. prop. 13.) the latus rectum L is equal to the quantity $\frac{QT^2}{QR}$ in its ultimate ſtate when the points P and Q coincide. But the lineola QR in a given time is as the generating centripetal force; that is (by ſuppoſition) reciprocally as SP^2. And therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$, that is, the latus rectum L is in the duplicate ratio of the area $QT \times SP$. *Q. E. D.*

COR. Hence the whole area of the ellipſis, and the rectangle under the axes, which is proportional to it, is in the ratio compounded of the ſubduplicate ratio of the latus rectum, and the ratio of the periodic time. For the whole area is as the area $QT \times SP$, deſcribed in a given time, multiplied by the periodic time.

PRO-

Plate V. Vol. I. P. 86.

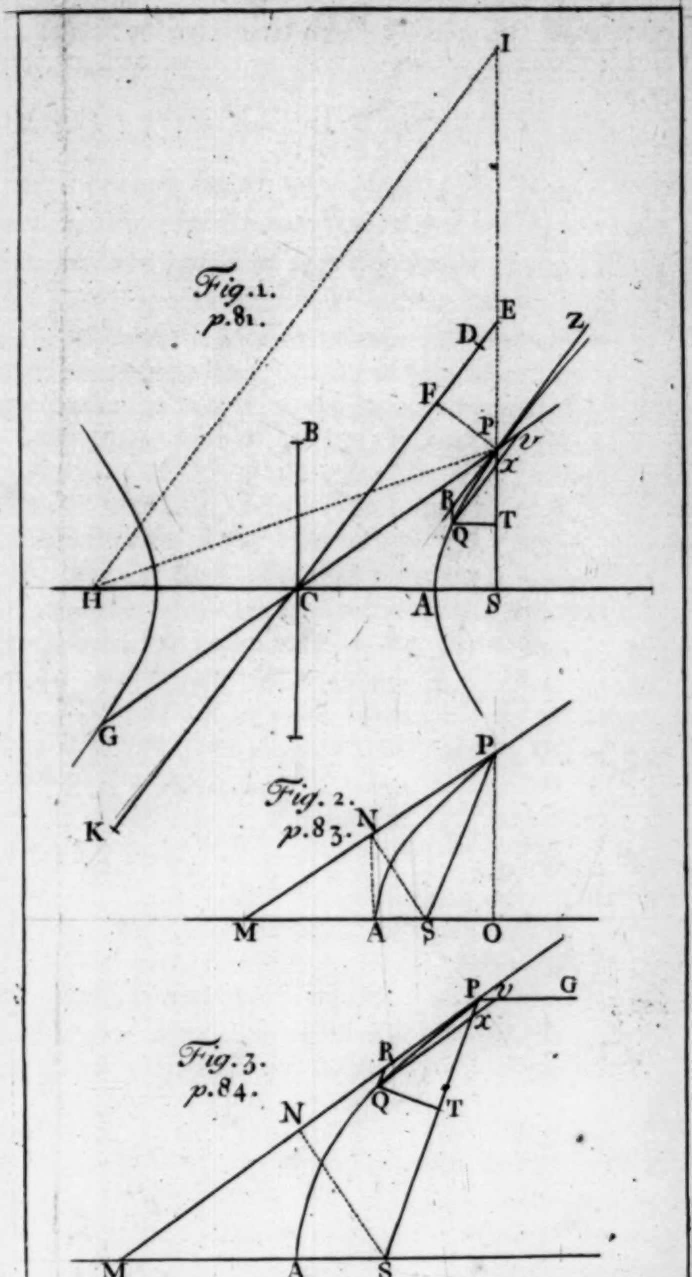


Fig. 1.
p. 81.

Fig. 2.
p. 83.

Fig. 3.
p. 84.

PROPOSITION XV. THEOREM VII.

The ſame things being ſuppoſed, I ſay that the periodic times in ellipſes are in the ſeſquiplicate ratio of their greater axes.

For the leſſer axe is a mean proportional between the greater axe and the latus rectum; and therefore the rectangle under the axes is in the ratio compounded of the ſubduplicate ratio of the latus rectum and the ſeſquiplicate ratio of the greater axe. But this rectangle (by cor. prop. 14.) is in a ratio compounded of the ſubduplicate ratio of the latus rectum and the ratio of the periodic time. Subduct from both ſides the ſubduplicate ratio of the latus rectum, and there will remain the ſeſquiplicate ratio of the greater axe, equal to the ratio of the periodic time. *Q. E. D.*

COR. Therefore the periodic times in ellipſes are the ſame as in circles whoſe diameters are equal to the greater axes of the ellipſes.

PROPOSITION XVI. THEOREM VIII.

The ſame things being ſuppoſed, and right lines being drawn to the bodies that ſhall touch the orbits, and perpendiculars being let fall on thoſe tangents from the common focus: I ſay that the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inverſely, and the ſubduplicate

cate ratio of the principal latera recta directly. Pl. 6. Fig. 2.

From the focus *S*, draw *ST* perpendicular to the tangent *PR*, and the velocity of the body *P* will be reciprocally in the ſubduplicate ratio of the quantity $\frac{ST^2}{L}$. For that velocity is as the infinitely ſmall arc *PQ* deſcribed in a given moment of time, that is, (by lem. 7.) as the tangent *PR*; that is, (becauſe of the proportionals *PR* to *QT* and *SP* to *ST*) as $\frac{SP \times QT}{ST}$, or as *ST* reciprocally and $SP \times QT$ directly; but $SP \times QT$ is as the area deſcribed in the given time, that is (by prop. 14.) in the ſubduplicate ratio of the latus rectum. *Q. E. D.*

Cor. 1. The principal latera recta are in a ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

Cor. 2. The velocities of bodies, in their greateſt and leaſt diſtances from the common focus, are in the ratio compounded of the ratio of the diſtances inverſely, and the ſubduplicate ratio of the principal latera recta directly. For thoſe perpendiculars are now the diſtances.

Cor. 3. And therefore the velocity in a conic ſection, at its greateſt or leaſt diſtance from the focus, is to the velocity in a circle at the ſame diſtance from the centre, in the ſubduplicate ratio of the principal latus rectum to the double of that diſtance.

Cor. 4. The velocities of the bodies revolving in ellipſes, at their mean diſtances from the common focus, are the ſame as thoſe of bodies revolving in circles, at the ſame diſtances; that is (by cor. 6. prop. 4.) reciprocally in the ſubduplicate ratio of the diſtances.

For the perpendiculars are now the leſſer ſemi-axes, and theſe are as mean proportionals between the diſtances and the latera recta. Let this ratio inverſely be compounded with the ſubduplicate ratio of the latera recta directly, and we ſhall have the ſubduplicate ratio of the diſtances inverſely.

COR. 5. In the ſame figure, or even in different figures, whoſe principal latera recta are equal, the velocity of a body is reciprocally as the perpendicular let fall from the focus on the tangent.

COR. 6. In a parabola, the velocity is reciprocally in the ſubduplicate ratio of the diſtance of the body from the focus of the figure; it is more variable in the ellipſis, and leſs in the hyperbola, than according to this ratio. For (by cor. 2. lem. 14.) the perpendicular let fall from the focus on the tangent of a parabola is in the ſubduplicate ratio of the diſtance. In the hyperbola the perpendicular is leſs variable, in the ellipſis more.

COR. 7. In a parabola, the velocity of a body at any diſtance from the focus, is to the velocity of a body revolving in a circle at the ſame diſtance from the centre, in the ſubduplicate ratio of the number 2 to 1; in the ellipſis it is leſs, and in the hyperbola greater, than according to this ratio. For (by cor. 2. of this prop.) the velocity at the vertex of a parabola is in this ratio, and (by cor. 6. of this prop. and prop. 4.) the ſame proportion holds in all diſtances. And hence alſo in a parabola, the velocity is every where equal to the velocity of a body revolving in a circle at half the diſtance; in the ellipſis it is leſs, and in the hyperbola greater.

Cor. 8. The velocity of a body revolving in any conic ſection is to the velocity of a body revolving in a circle at the diſtance of half the principal latus rectum of the ſection, as that diſtance to the perpendicular let fall from the focus on the tangent of the ſection. This appears from cor. 5.

Cor. 9. Wherefore ſince (by cor. 6. prop. 4.) the velocity of a body revolving in this circle is to the velocity of another body revolving in any other circle, reciprocally in the ſubduplicate ratio of the diſtances; therefore *ex æquo* the velocity of a body revolving in a conic ſection will be to the velocity of a body revolving in a circle at the ſame diſtance, as a mean proportional between that common diſtance and half the principal latus rectum of the ſection, to the perpendicular let fall from the common focus upon the tangent of the ſection.

PROPOSITION XVII. PROBLEM IX.

Suppoſing the centripetal force to be reciprocally proportional to the ſquares of the diſtances of places from the centre, and that the abſolute quantity of that force is known; it is required to determine the line, which a body will deſcribe that is let go from a given place with a given velocity in the direction of a given right line.

Let the centripetal force tending to the point *S* (*Pl. 6. Fig.* 3.) be ſuch, as will make the body *p* revolve in any given orbit *pq*; and ſuppoſe the velocity of this body in the place *p* is known. Then from the place *P*, ſuppoſe the body *P* to be let go with a given velocity in the direction of the line *PR*; but by virtue of a centripetal force to be immediately turned aſide from that right line into the conic ſection *PQ*. This the right line *PR* will therefore touch in *P*. Suppoſe likewiſe that the right line *pr* touches the orbit *pq* in *p*; and if from *S* you ſuppoſe perpendiculars let fall on thoſe tangents, the principal latus rectum of the conic ſection (by cor. 1. prop. 16.) will be to the principal latus rectum of that orbit, in a ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities; and is therefore given. Let this latus rectum be *L*. The focus *S* of the conic ſection is alſo given. Let the angle *RPH* be the complement of the angle *RPS* to two right; and the line *PH*, in which the other focus *H* is placed, is given by poſition. Let fall *SK* perpendicular on *PH*, and erect the conjugate ſemi-axe *BC*; this done, we ſhall have $SP^2 - 2KPH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 = \overline{SP + PH}^2 - L \times \overline{SP + PH} = SP^2 + 2SPH + PH^2 - L \times \overline{SP + PH}$. Add on both ſides $2KPH - SP^2 - PH^2 + L \times \overline{SP + PH}$, and we ſhall have $L \times \overline{SP + PH} = 2SPH + 2KPH$, or $SP + PH$ to PH as $2SP + 2KP$ to L. Whence *PH* is given both in length and poſition. That is, if the velocity of the body in *P* is ſuch that the latus rectum *L* is leſs than $2SP + 2KP$, *PH* will lie on the ſame ſide of the tangent *PR* with the line *SP*; and therefore the figure will be an ellipſis, which from the given foci *S*, *H*, and the principal axe $SP + PH$, is given alſo. But if the velocity of the body is ſo great, that the latus rectum *L* becomes equal to $2PS + 2KP$, the length *PH* will be infinite; and therefore the figure will be a parabola, which has its axe *SH* parallel to the line PK, and is thence given. But if the body goes from its place P with a yet greater velocity, the length PH is to be taken on the other ſide the tangent; and ſo the tangent paſſing between the foci, the figure will be an hyperbola having its principal axe equal to the difference of the lines SP and PH, and thence is given. For if the body, in theſe caſes, revolves in a conic ſection ſo found, it is demonſtrated in prop. 11, 12, and 13. that the centripetal force will be reciprocally as the ſquare of the diſtance of the body from the centre of force S; and therefore we have rightly determined the line PQ, which a body let go from a given place P with a given velocity, and in the direction of the right line PR given by poſition, would deſcribe with ſuch a force. *Q.E.F.*

Cor. 1. Hence in every conic ſection, from the principal vertex D, the latus rectum L, and the focus S given, the other focus H is given, by taking DH to DS as the latus rectum to the difference between the latus rectum and $4DS$. For the proportion, $SP+PH$ to PH as $2PS+2KP$ to L, becomes, in the caſe of this corollary, $DS+DH$ to DH as $4DS$ to L, and by diviſion DS to DH as $4DS-L$ to L.

Cor. 2. Whence if the velocity of a body in the principal vertex D is given, the orbit may be readily found; to wit, by taking its latus rectum to twice the diſtance DS, in the duplicate ratio of this given velocity to the velocity of a body revolving in a circle at the diſtance DS (by cor. 3. prop. 16.) and then taking DH to DS as the latus rectum to the difference between the latus rectum and $4DS$.

Cor. 3. Hence alſo if a body move in any conic ſection, and is forced out of its orbit by any impulſe; you may diſcover the orbit in which it will afterwards purſue its courſe. For by compounding the proper motion of the body with that motion, which the impulſe alone would generate, you'll have the motion with which the body will go off from a given place of impulſe, in the direction of a right line given in poſition.

Cor. 4. And if that body is continually diſturbed by the action of ſome foreign force, we may nearly know its courſe, by collecting the changes which that force introduces in ſome points, and eſtimating the continual changes it will undergo in the intermediate places, from the analogy that appears in the progreſs of the ſeries.

SCHOLIUM.

If a body *P* (*Pl. 6. Fig.* 4.) by means of a centripetal force tending to any given point *R* move in the perimeter of any given conic ſection, whoſe centre is *C*; and the law of the centripetal force is required: Draw *C G* parallel to the radius *R P*, and meeting the tangent *P G* of the orbit in *G*; and the force required (by cor. 1. & ſchol. prop. 10. & cor. 3. prop. 7.) will be as $\frac{CG^3}{RP^2}$.

* * *

SECTION IV.

Of the finding of elliptic, parabolic, and hyperbolic orbits, from the focus given.

LEMMA XV.

If from the two foci S, H, (Pl. 7. Fig. 1.) *of any ellipſis or hyperbola, we draw to any third point* V *the right lines* S V, H V, *whereof one* H V *is equal to the principal axis of the figure, that is, to the axis in which the foci are ſituated, the other* S V *is biſected in* T *by the perpendicular* T R *let fall upon it; that perpendicular* T R *will ſomewhere touch the conic ſection: and* vice verſa, *if it does touch it,* H V *will be equal to the principal axis of the figure.*

For, let the perpendicular *T R* cut the right line *H V*, produced if need be, in *R*; and join *S R*. Becauſe

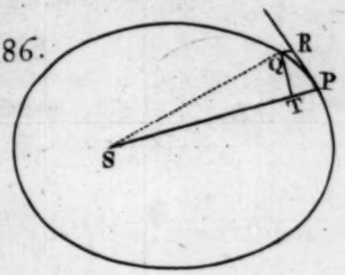


Fig. 1. p. 86.

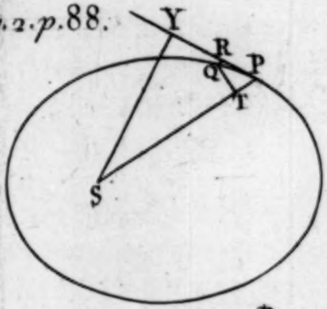


Fig. 2. p. 88.

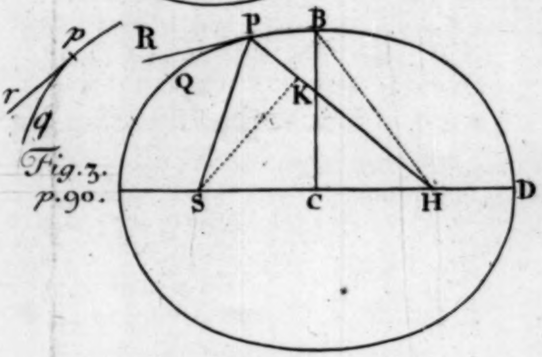


Fig. 3. p. 90.

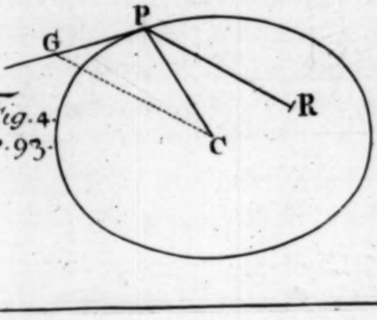


Fig. 4. p. 93.

cauſe *TS*, *TV* are equal, therefore the right lines *SR*, *VR*, as well as the angles *TRS*, *TRV*, will be alſo equal. Whence the point *R* will be in the conic ſection, and the perpendicular *TR* will touch the ſame: and the contrary. *Q. E. D.*

PROPOSITION XVIII. PROBLEM X.

From a focus and the principal axes given, to deſcribe elliptic and hyperbolic trajectories, which ſhall paſs through given points, and touch right lines given by poſition. Pl. 7. Fig. 2.

Let *S* be the common focus of the figures; *AB* the length of the principal axis of any trajectory; *P* a point through which the trajectory ſhould paſs; and *TR* a right line which it ſhould touch. About the centre *P*, with the interval *AB*—*SP*, if the orbit is an ellipſis, or *AB*+*SP* if the orbit is an hyperbola, deſcribe the circle *HG*. On the tangent *TR* let fall the perpendicular *ST*, and produce the ſame to *V*, ſo that *TV* may be equal to *ST*; and about *V* as a centre with the interval *AB* deſcribe the circle *FH*. In this manner whether two points *P*, *p*, are given, or two tangents *TR*, *tr*, or a point *P* and a tangent *TR*, we are to deſcribe two circles. Let *H* be their common interſection, and from the foci *S*, *H* with the given axis deſcribe the trajectory. I ſay the thing is done. For (becauſe *PH*+*SP* in the ellipſis, and *PH*—*SP* in the hyperbola is equal to the axis) the deſcribed trajectory will paſs through the point *P*, and (by the preceding lemma) will touch the right line *TR*. And by the ſame argument it will either paſs through the two points *P*, *p*, or touch the two right lines *TR*, *tr*. *Q. E. F.*

PROPOSITION XIX. PROBLEM XI.

About a given focus, to deſcribe a parabolic trajectory, which ſhall paſs through given points, and touch right lines given by poſition. Pl. 7. Fig. 3.

Let *S* be the focus, *P* a point, and *TR* a tangent of the trajectory to be deſcribed. About *P* as a centre, with the interval *PS*, deſcribe the circle *FG*. From the focus let fall *ST* perpendicular on the tangent, and produce the ſame to *V*, ſo as *TV* may be equal to *ST*. After the ſame manner another circle *fg* is to be deſcribed, if another point *p* is given; or another point *v* is to be found, if another tangent *tr* is given; then draw the right line *IF*, which ſhall touch the two circles *FG*, *fg*, if two points *P*, *p* are given, or paſs through the two points *V*, *v*, if two tangents *TR*, *tr* are given, or touch the circle *FG* and paſs through the point *V*, if the point *P* and the tangent *TR* are given. On *FI* let fall the perpendicular *SI*, and biſect the ſame in *K*; and with the axis *SK*, and principal vertex *K* deſcribe a parabola. I ſay the thing is done. For this parabola (becauſe *SK* is equal to *IK*, and *SP* to *FP*) will paſs through the point *P*; and (by cor. 3. lem. 14.) becauſe *ST* is equal to *TV*, and *STR* a right angle, it will touch the right line *TR*. *Q. E. F.*

PROP.

PROPOSITION XX. PROBLEM XII.

About a given focus to deſcribe any trajectory given in ſpecie, which ſhall paſs thro' given points and touch right lines given by poſition.

CASE 1. About the focus *S* (*Pl.* 7. *Fig.* 4.) it is required to deſcribe a trajectory *ABC*, paſſing thro' two points *B*, *C*. Becauſe the trajectory is given in ſpecie, the ratio of the principal axe to the diſtance of the foci will be given. In that ratio take *KB* to *BS* and *LC* to *CS*. About the centres *B*, *C*, with the intervals *BK*, *CL* deſcribe two circles, and on the right line *KL*, that touches the ſame in *K* and *L*, let fall the perpendicular *SG*; which cut in *A* and *a*, ſo that *GA* may be to *AS*, and *Ga* to *aS*, as *KB* to *BS*; and with the axe *Aa*, and vertices *A*, *a*, deſcribe a trajectory. I ſay the thing is done. For let *H* be the other focus of the deſcribed figure, and ſeeing *GA* is to *AS* as *Ga* to *aS*, then by diviſion we ſhall have *Ga*—*GA* or *Aa* to *aS*—*AS* or *SH* in the ſame ratio, and therefore in the ratio which the principal axe of the figure to be deſcribed has to the diſtance of its foci; and therefore the deſcribed figure is of the ſame ſpecies with the figure which was to be deſcribed. And ſince *KB* to *BS*, and *LC* to *CS* are in the ſame ratio, this figure will paſs thro' the points *B*, *C*, as is manifeſt from the conic ſections.

CASE 2. About the focus *S* (*Pl.* 7. *Fig.* 5.) it is required to deſcribe a trajectory, which ſhall ſomewhere touch two right lines *TR*, *tr*. From the focus on thoſe tangents let fall the perpendiculars *ST*, *St*, which produce to *V*, *v*, ſo that *TV*, *tv* may be equal to *TS*, *tS*. Biſect *Vv* in *O*, and erect the indefinite perpendicular *OH*, and cut the right Line *VS* infinitely produced in *K* and *k*, ſo that *VK* be to *KS*, and *Vk* to *kS* as the principal axe of the trajectory to be deſcribed is to the diſtance of it's foci. On the diameter *Kk* deſcribe a circle cutting *OH* in *H*; and with the foci *S*, *H*, and principal axe equal to *VH*, deſcribe a trajectory. I ſay the thing is done. For, biſecting *Kk* in *X*, and joining *HX*, *HS*, *HV*, *Hv*, becauſe *VK* is to *KS*, as *Vk* to *kS*; and by compoſition, as *VK* + *Vk* to *KS* + *kS*; and by diviſion, as *Vk* — *VK* to *kS* − *KS*, that is, as 2 *VX* to 2 *KX* and 2 *KX* to 2 *SX*, and thereſore as *VX* to *HX* and *HX* to *SX*, the triangles *VXH*, *HXS* will be ſimilar; Therefore *VH* will be to *SH*, as *VX* to *XH*; and therefore as *VK* to *KS*. Whereſore *VH* the principal axe of the deſcribed trajectory has the ſame ratio to *SH* the diſtance of the foci, as the principal axe of the trajectory which was to be deſcribed has to the diſtance of its foci; and is thereſore of the ſame ſpecies. And ſeeing *VH*, *vH*, are equal to the principal axe, and *VS*, *vS* are perpendicularly biſected by the right lines *TR*, *tr*; 'tis evident (by lem. 15.) that thoſe right lines touch the deſcribed trajectory. *Q. E. F.*

Case 3. About the focus *S* (*Pl.* 7. *Fig.* 6.) it is required to deſcribe a trajectory, which ſhall touch a right line *TR* in a given point *R*. On the right line *TR* let fall the perpendicular *ST*, which produce to *V*, ſo that *TV* may be equal to *ST*, join *VR*, and cut the right line *VS* indefinitely produced in *K* and *k*, ſo that *VK* may be to *SK*, and *Vk* to *Sk* as the principal axe of the ellipſis to be deſcribed, to the diſtance of its foci; and on the diameter *Kk* deſcribing a circle, cut the right line *VR* produced in *H*, then with the foci *S*, *H*, and principal axe equal to *VH*, deſcribe a trajectory. I ſay the thing is done. For *VH* is to *SH* as *VK* to *SK*, and therefore as the principal axe of the trajectory which was to be deſcribed to the diſtance of its foci, (as appears from what we have demonſtrated in Caſe 2.) and therefore the deſcribed trajectory is of the ſame ſpecies with that which was to be deſcribed; but that the right line *TR*, by which the angle *VRS* is biſected, touches the trajectory in the point *R*, is certain from the properties of the conic ſections. *Q.E.F.*

Case 4. About the focus *S* (*Pl.* 7. *Fig.* 7.) it is required to deſcribe a trajectory *APB* that ſhall touch a right line *TR*, and paſs thro' any given point *P* without the tangent, and ſhall be ſimilar to the figure *apb*, deſcribed with the principal axe *ab*, and foci *s*, *h*. On the tangent *TR* let fall the perpendicular *ST*; which produce to *V*, ſo that *TV* may be equal to *ST*. And making the angles *hsq*, *shq* equal to the angles *VSP*, *SVP*; about *q* as a centre, and with an interval which ſhall be to *ab* as *SP* to *VS* deſcribe a circle cutting the figure *apb* in *p*: join *sp*, and draw *SH*, ſuch that it may be to *sh*, as *SP* is to *sp*, and may make the angle *PSH* equal to the angle *psh*, and the angle *VSH* equal to the angle *psq*. Then with the foci *S*, *H*, and principal axe *AB* equal to the diſtance *VH*, deſcribe a conic ſection. I ſay the thing is done. For if *sv* is drawn ſo that it ſhall be to *sp* as *sh* is to *sq*, and ſhall make the angle *vsp* equal to the angle *hsq*, and the angle *vsh* equal to the angle *psq*, the triangles *svh*, *spq*, will be ſimilar, and therefore *vh* will be to *pq*, as *sh* is to *sq*, that is, (becauſe of the ſimilar triangles *VSP*, *hsq*) as *VS* is to *SP* or as *ab* to *pq*. Wherefore *v h* and *a b* are equal. But becauſe of the ſimilar triangles *VSH*, *v s h*, *VH* is to *S H* as *v h* to *s h*; that is, the axe of the conic ſection now deſcribed is to the diſtance of its foci, as the axe *a b* to the diſtance of the foci *s*, *h*; and therefore the figure now deſcribed is ſimilar to the figure *a p b*. But, becauſe the triangle *P S H* is ſimilar to the triangle *p s h*, this figure paſſes through the point *P*; and becauſe *VH* is equal to its axis, and *V S* is perpendicularly biſected by the right line *TR*, the ſaid figure touches the right line *T R*. *Q. E. F.*

LEMMA XVI.

From three given points to draw to a fourth point that is not given three right lines whoſe differences ſhall be either given or none at all.

CASE 1. Let the given points be *A*, *B*, *C*, (*Pl.* 8. *Fig.* 1.) and *Z* the fourth point which we are to find; becauſe of the given difference of the lines *A Z*, *B Z*, the locus of the point *Z* will be an hyperbola, whoſe foci are *A* and *B*, and whoſe principal axe is the given difference. Let that axe be *M N*. Taking *PM* to *MA*, as *M N* is to *A B*, erect *P R* perpendicular to *AB*, and let fall *ZR* perpendicular to *PR*; then, from the nature of the hyperbola, *ZR* will be to *A Z* as *MN* is to *AB*. And by the like argument, the locus of the point *Z* will be another hyperbola, whoſe foci are *A*, *C*, and whoſe principal axe is the difference between *A Z* and *C Z*; and *Q S* a perpendicular on *A C* may be drawn, to which (*Q S*) if from any point *Z* of this hyperbola

Fig. 1. p. 94.

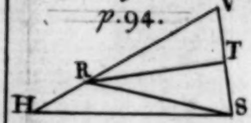


Fig. 2. p. 95.

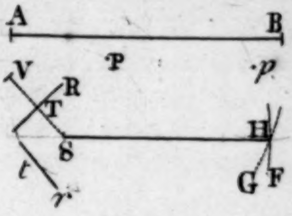


Fig. 3. p. 96.

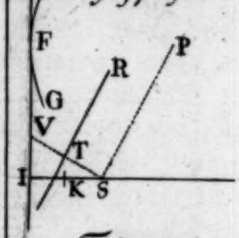


Fig. 4. p. 97.

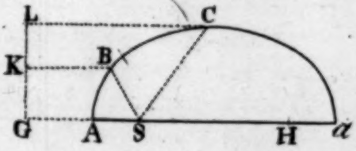


Fig. 5. ib.

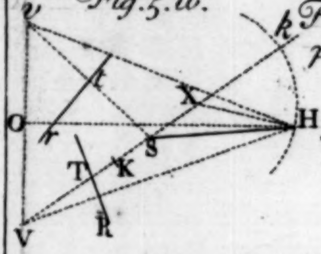


Fig. 6. p. 98.

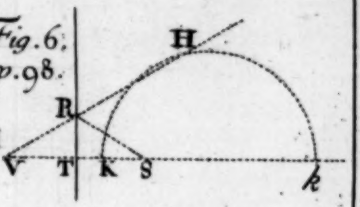
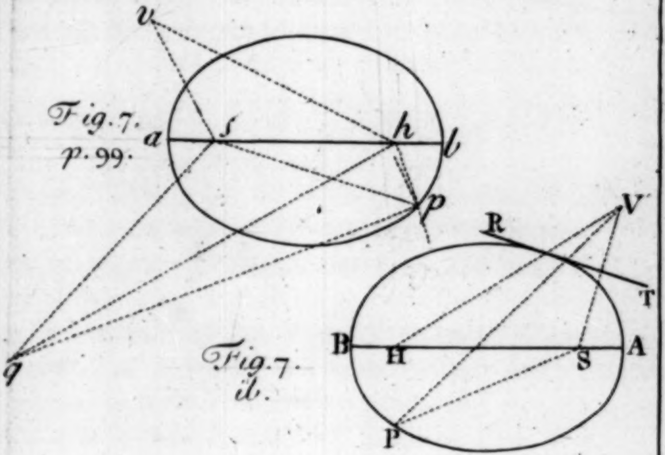


Fig. 7. p. 99.

Fig. 7. ib.



bola a perpendicular ZS is let fall, this (ZS) ſhall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratio's of ZR and ZS to AZ are given, and conſequently the ratio of ZR to ZS one to the other; and therefore if the right lines RP, SQ meet in T, and TZ and TA are drawn, the figure $TRZS$ will be given in ſpecie, and the right line TZ, in which the point Z is ſomewhere placed, will be given in poſition. There will be given alſo the right line TA, and the angle ATZ; and becauſe the ratio's of AZ and TZ to ZS are given, their ratio to each other is given alſo; and thence will be given likewiſe the triangle ATZ whoſe vertex is the point Z. *Q. E. I.*

CASE 2. If two of the three lines, for example AZ and BZ, are equal, draw the right line TZ ſo as to biſect the right line AB; then find the triangle ATZ as above. *Q. E. I.*

CASE 3. If all the three are equal, the point Z will be placed in the centre of a circle that paſſes thro' the points A, B, C. *Q. E. I.*

This problematic lemma is likewiſe ſolved in *Apollonius*'s Book of Tactions reſtored by *Vieta.*

PROPOSITION XXI. PROBLEM XIII.

About a given focus to deſcribe a trajectory, that ſhall paſs through given points and touch right lines given by poſition.

Let the ſocus S, (*Pl.* 8. *Fig.* 2.) the point P, and the tangent TR be given, and ſuppoſe that the other focus H is to be found. On the tangent let fall the perpendicular ST, which produce to Y, ſo that TY

may be equal to *ST*, and *YH* will be equal to the principal axe. Join *SP*, *HP*, and *SP* will be the difference between *HP* and the principal axe. After this manner if more tangents *TR* are given, or more points *P*, we ſhall always determine as many lines *YH* or *PH*, drawn from the ſaid points *Y* or *P*, to the focus *H*, which either ſhall be equal to the axes, or differ from the axes by given lengths *SP*; and therefore which ſhall either be equal among themſelves, or ſhall have given differences; from whence (by the preceding lemma) that other focus *H* is given. But having the foci and the length of the axe (which is either *YH*; or, if the trajectory be an ellipſis, $PH + SP$, or $PH - SP$ if it be an hyperbola) the trajectory is given. *Q. E. I.*

SCHOLIUM.

When the trajectory is an hyperbola, I do not comprehend its conjugate hyperbola under the name of this trajectory. For a body going on with a continued motion can never paſs out of one hyperbola into its conjugate hyperbola.

The caſe when three points are given is more readily ſolved thus. Let *B*, *C*, *D* (*Pl.* 8. *Fig.* 3.) be the given points. Join *BC*, *CD*, and produce them to *E*, *F*; ſo as *EB* may be to *EC*, as *SB* to *SC*; and *FC* to *FD*, as *SC* to *SD*. On *EF* drawn and produced let fall the perpendiculars *SG*, *BH*, and in *GS* produced indefinitely take *GA* to *AS*, and *Ga* to *aS*, as *HB* is to *BS*; then *A* will be the vertex, and *Aa* the principal axe of the trajectory: Which, according as *GA* is greater than, equal to, or leſs than *AS*, will be either an ellipſis, a parabola or an hyperbola; the point a in the firſt caſe falling on the ſame ſide of the line GF as the point A; in the ſecond, going off to an infinite diſtance; in the third, falling on the other ſide of the line GF. For if on GF, the perpendiculars CI, DK are let fall, IC will be to HB as EC to EB; that is, as SC to SB; and by permutation IC to SC as HB to SB, or as GA to SA. And, by the like argument, we may prove that KD is to SD in the ſame ratio. Wherefore the points B, C, D lie in a conic ſection deſcribed about the focus S, in ſuch manner that all the right lines drawn from the focus S to the ſeveral points of the ſection, and the perpendiculars let fall from the ſame points on the right line GF are in that given ratio.

That excellent geometer M. *De la Hire* has ſolved this problem much after the ſame way in his conics, prop. 25. lib. 8.

SECTION V.

How the orbits are to be found when neither focus is given.

LEMMA XVII.

If from any point P *of a given conic ſection, to the four produced ſides* AB, CD, AC, DB *of any trapezium* ABDC *inſcribed in that ſection, as many right lines* PQ, PR, PS, PT *are drawn in given angles, each line to each ſide; the rectangle* PQ×PR *of thoſe on the oppoſite ſides* AB, CD, *will be to the rectangle* PS×PT *of thoſe on the other two oppoſite ſides* AC, BD, *in a given ratio.*

CASE 1. Let us ſuppoſe firſt that the lines drawn to one pair of oppoſite ſides are parallel to either of the other ſides; as *PQ* and *PR* (*Pl.* 8. *Fig.* 4) to the ſide *AC*, and *PS* and *PT* to the ſide *AB*. And farther, that one pair of the oppoſite ſides, as *AC* and *BD*, are parallel betwixt themſelves; then the right line which biſects thoſe parallel ſides will be one of the diameters of the conic ſection, and will likewiſe biſect *RQ*. Let *O* be the point in which *RQ* is biſected, and *PO* will be an ordinate to that diameter. Produce *PO* to *K*, ſo that *OK* may be equal to *PO*, and *OK* will be an ordinate on the other ſide of that diameter. Since therefore the points *A*, *B*, *P*, and *K* are placed in the conic ſection, and *PK* cuts *AB* in a given angle, the rectangle *PQK* (by prop. 17. 19. 21. & 23. book 3. of *Apollonius*'s conics) will be to the rectangle *AQB* in a given ratio. But *QK* and *PR* are equal, as being the differences of the equal lines *OK*, *OP*, and *OQ*, *OR*; whence the rectangles *PQK* and $PQ \times PR$ are equal; and therefore the rectangle $PQ \times PR$ is to the rectangle *AQB*, that is, to the rectangle $PS \times PT$ in a given ratio. *Q. E. D.*

CASE 2. Let us next ſuppoſe that the oppoſite ſides *AC* and *BD* (*Pl.* 8. *Fig.* 5.) of the trapezium, are not parallel. Draw *Bd* parallel to *AC* and meeting as well the right line *ST* in *t*, as the conic ſection in *d*. Join *Cd* cutting *PQ* in *r*, and draw *DM* parallel to *PQ*, cutting *Cd* in *M* and *AB* in *N*. Then (becauſe of the ſimilar triangles *BTt*, *DBN*,) *Bt* or *PQ* is to *Tt* as *DN* to *NB*. And ſo *Rr* is to *AQ* or *PS* as *DM* to *AN*. Wherefore, by multiplying the antecedents by the antecedents and the conſequents by the conſequents, as the rectangle $PQ \times Rr$ is to the rectangle $PS \times Tt$, ſo will the rectangle *NDM* be to the rectangle *ANB*, and (by caſe 1.) ſo is the rectangle $PQ \times Pr$ to the rectangle $PS \times Pt$, and by diviſion, ſo is the rectangle $PQ \times PR$ to the rectangle $PS \times PT$. *Q. E. D.*

CASE 3. Let us ſuppoſe laſtly the four lines *PQ*, *PR*, *PS*, *PT* (*Pl.* 8. *Fig.* 6.) not to be parallel to the ſides *AC*, *AB*, but any way inclined to them. In their place draw *Pq*, *Pr* parallel to *AC*; and *Ps*, *Pt* parallel to *AB*; and becauſe the angles of the triangles *PQq*, *PRr*, *PSs*, *PTt* are given, the ratio's of *PQ* to *Pq*, *PR* to *Pr*, *PS* to *Ps*, *PT* to *Pt* will be alſo given; and therefore the compounded ratio's *PQ* × *PR* to *Pq* × *Pr*, and *PS* × *PT* to *Ps* × *Pt* are given. But from what we have demonſtrated before, the ratio of *Pq* × *Pr* to *Ps* × *Pt* is given; and therefore alſo the ratio of *PQ* × *PR* to *PS* × *PT*. *Q.E.D.*

LEMMA XVIII.

The ſame things ſuppoſed, if the rectangle PQ × PR *of the lines drawn to the two oppoſite ſides of the trapezium is to the rectangle* PS × PT *of thoſe drawn to the other two ſides, in a given ratio; the point* P, *from whence thoſe lines are drawn, will be placed in a conic ſection deſcribed about the trapezium.* (Pl. 8. Fig. 7.)

Conceive a conic ſection to be deſcribed paſſing through the points *A*, *B*, *C*, *D*, and any one of the infinite number of points *P*, as for example *p*; I ſay the point *P* will be always placed in this ſection. If you deny the thing, join *AP* cutting this conic ſection ſomewhere elſe if poſſible than in *P*, as in *b*. Therefore if from thoſe points *p* and *b*, in the given angles to the ſides of the trapezium, we draw the right ines *pq*, *pr*, *ps*, *pt*, and *bk*, *bn*, *bf*, *bd*, we ſhall have as $bk \times bn$ to $bf \times bd$, ſo (by Lem. 17.) $pq \times pr$ to $ps \times pt$; and ſo (by ſuppoſition) $PQ \times PR$ to $PS \times PT$. And becauſe of the ſimilar trapezia $bkAf$, $PQAS$, as bk to bf, ſo PQ to PS. Wherefore by dividing the terms of the preceding proportion by the correſpondent terms of this, we ſhall have bn to bd as PR to PT. And therefore the equiangular trapezia $Dnbd$, $DRPT$ are ſimilar, and conſequently their diagonals Db, DP do coincide. Wherefore b falls in the interſection of the right lines AP, DP, and conſequently coincides with the point P. And therefore the point P where-ever it is taken, falls to be in the aſſigned conic ſection. *Q.E.D.*

COR. Hence if three right lines PQ, PR, PS, are drawn from a common point P to as many other right lines given in poſition AB, CD, AC, each to each, in as many angles reſpectively given, and the rectangle $PQ \times PR$ under any two of the lines drawn be to the ſquare PS^2 of the third in a given ratio: The point P, from which the right lines are drawn, will be placed in a conic ſection that touches the lines AB, CD in A and C; and the contrary. For the poſition of the three right lines AB, CD, AC remaining the ſame, let the line BD approach to and coincide with the line AC; then let the line PT come likewiſe to coincide with the line PS; and the rectangle $PS \times PT$ will become PS^2, and the right lines AB, CD, which before did cut the curve in the points A and B, C and D, can no longer cut, but only touch, the curve in thoſe co-inciding points.

SCHOLIUM.

In this lemma, the name of conic ſection is to be underſtood in a large ſenſe, comprehending as well the rectilinear ſection thro' the vertex of the cone, as the circular one parallel to the baſe. For if the point *p* happens to be in a right line, by which the points *A* and *D* or *C* and *B* are joined, the conic ſection will be changed into two right lines, one of which is that right line upon which the point *p* falls, and the other is a right line that joins other two of the four points. If the two oppoſite angles of the trapezium taken together are equal to two right angles, and if the four lines *PQ*, *PR*, *PS*, *PT* are drawn to the ſides thereof at right angles, or any other equal angles, and the rectangle $PQ \times PR$ under two of the lines drawn *PQ* and *PR*, is equal to the rectangle $PS \times PT$ under the other two *PS* and *PT*, the conic ſection will become a circle. And the ſame thing will happen, if the four lines are drawn in any angles, and the rectangle $PQ \times PR$ under one pair of the lines drawn, is to the rectangle $PS \times PT$ under the other pair, as the rectangle under the ſines of the angles *S*, *T*, in which the two laſt lines *PS*, *PT* are drawn, to the rectangle under the ſines of the angles *Q*, *R*, in which the two firſt *PQ*, *PR* are drawn. In all other caſes the locus of the point *P* will be one of the three figures, which paſs commonly by the name of the conic ſections. But in room of the trapezium *ABCD*, we may ſubſtitute a quadrilateral figure whoſe two oppoſite ſides croſs one another like diagonals. And one or two of the four points *A*, *B*, *C*, *D* may be ſuppoſed to be removed to an infinite diſtance, by which means the ſides of the figure which converge to thoſe points, will become parallel: And in this caſe the conic ſection will paſs through the other points, and will go the ſame way as the parallels *in infinitum*.

LEMMA XIX.

To find a point P (Pl. 8. Fig. 8.) *from which if four right lines* PQ, PR, PS, PT *are drawn to as many other right lines* AB, CD, AC, BD *given by poſition, each to each, at given angles, the rectangle* PQ×PR, *under any two of the lines drawn, ſhall be to the rectangle* PS×PT, *under the other two, in a given ratio.*

Suppoſe the lines *AB*, *CD*, to which the two right lines *PQ*, *PR*, containing one of the rectangles, are drawn to meet two other lines, given by poſition, in the points *A*, *B*, *C*, *D*. From one of thoſe as *A*, draw any right line *AH*, in which you would find the point *P*. Let this cut the oppoſite lines *BD*, *CD*, in *H*, and *I*; and, becauſe all the angles of the figure are given, the ratio of *PQ* to *PA*, and *PA* to *PS*, and therefore of *PQ* to *PS* will be alſo given. Subducting this ratio from the given ratio of *PQ*×*PR* to *PS*×*PT*, the ratio of *PR* to *PT* will be given; and adding the given ratio's of *PI* to *PR*, and *PT* to *PH*, the ratio of *PI* to *PH*, and therefore the point *P* will be given. *Q. E. I.*

COR. 1. Hence alſo a tangent may be drawn to any point *D* of the locus of all the points *P*. For the chord *PD*, where the points *P* and *D* meet, that is, where *AH* is drawn thro' the point *D*, becomes a tangent. In which caſe the ultimate ratio of the evaneſcent lines *IP* and *PH* will be found as above. Therefore draw *CF* parallel to *AD*, meeting *BD* in *F*, and cut it in *E* in the ſame ultimate ratio, then *DE* will be the tangent; becauſe *CF*, and the evaneſcent *IH* are parallel, and ſimilarly cut in *E* and *P*.

COR. 2. Hence alſo the locus of all the points *P* may be determined. Through any of the points *A*, *B*, *C*, *D*, as *A*, (*Pl.* 9. *Fig.* 1.) draw *AE* touching the locus, and through any other point *B* parallel to the tangent, draw *BF* meeting the locus in *F*: And find the point *F* by this lemma. Biſect *BF* in *G*, and drawing the indefinite line *AG*, this will be the poſition of the diameter to which *BG*, and *FG* are ordinates. Let this *AG* meet the locus in *H*, and *AH* will be its diameter or latus tranſverſum, to which the latus rectum will be as BG^2 to $AG \times GH$. If *AG* no where meets the locus, the line *AH* being infinite the locus will be a parabola; and its latus rectum correſponding to the diameter *AG* will be $\frac{BG^2}{AG}$. But if it does meet it any where, the locus will be an hyperbola, when the points *A* and *H* are placed on the ſame ſide the point *G*; and an ellipſis, if the point *G* falls between the points *A* and *H*; unleſs perhaps the angle *AGB* is a right angle, and at the ſame time BG^2 equal to the rectangle *AGH*, in which caſe the locus will be a circle.

And ſo we have given in this corollary a ſolution of that famous problem of the ancients concerning four lines, begun by *Euclid*, and carried on by *Apollonius*; and this not an analytical calculus, but a geometrical compoſition, ſuch as the ancients required.

LEMMA

Fig. 1. p. 100.

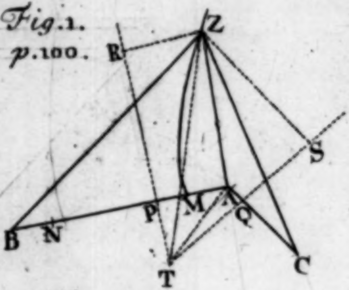


Fig. 2. p. 101.

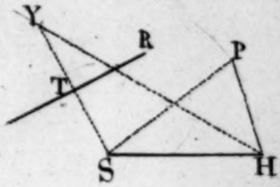


Fig. 3. p. 102.

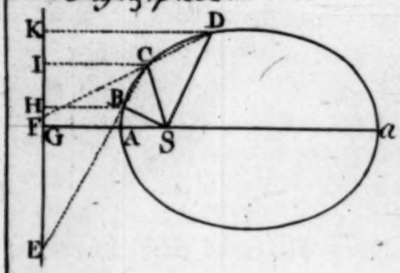


Fig. 4. p. 104.

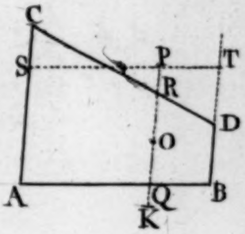


Fig. 5. p. 105.

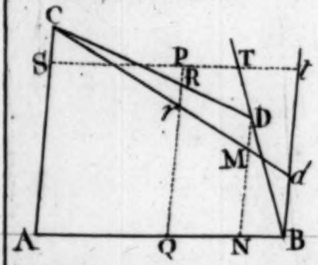


Fig. 6. ib.

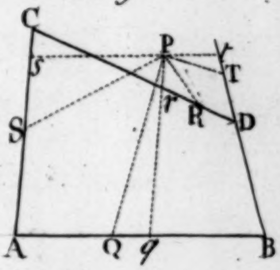


Fig. 7. p. 106.

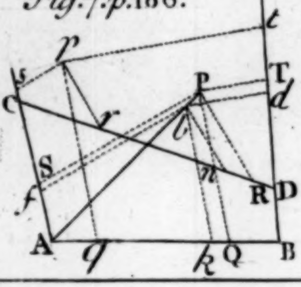
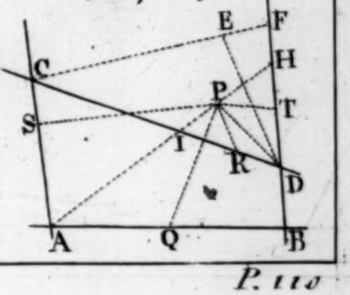


Fig. 8. p. 109.



LEMMA XX.

If the two oppoſite angular points A *and* P (Pl. 9. Fig. 2.) *of any parallelogram* ASPQ *touch any conic ſection in the points* A *and* P; *and the ſides* AQ, AS *of one of thoſe angles, indefinitely produced, meet the ſame conic ſection in* B *and* C; *and from the points of concourſe* B *and* C *to any fifth point* D *of the conic ſection, two right lines* BD, CD *are drawn meeting the two other ſides* PS, PQ *of the parallelogram, indefinitely produced, in* T *and* R; *the parts* PR *and* PT, *cut off from the ſides, will always be one to the other in a given ratio. And* vice verſa, *if thoſe parts cut off are one to the other in a given ratio, the locus of the point* D *will be a conic ſection, paſſing through the four points* A, B, C, P.

CASE 1. Join BP, CP, and from the point D draw the two right lines DG, DE, of which the firſt DG ſhall be parallel to AB, and meet PB, PQ, CA in H, I, G; and the other DE ſhall be parallel to AC, and meet PC, PS, AB, in F, K, E; and (by Lem. 17.) the rectangle $DE \times DF$ will be to the rectangle $DG \times DH$, in a given ratio. But PQ is to DE (or IQ) as PB to HB, and conſequently as PT to DH; and by permutation, PQ

is to PT, as DE to DH. Likewiſe PR is to DF as RC to DC, and therefore as (IG or) PS to DG; and, by permutation, PR is to PS as DF to DG; and, by compounding thoſe ratio's, the rectangle $PQ \times PR$ will be to the rectangle $PS \times PT$ as the rectangle $DE \times DF$ is to the rectangle $DG \times DH$, and conſequently in a given ratio. But PQ and PS are given, and therefore the ratio of PR to PT is given. *Q. E. D.*

CASE 2. But if PR and PT are ſuppoſed to be in a given ratio one to the other, then by going back again by a like reaſoning, it will follow that the rectangle $DE \times DF$ is to the rectangle $DG \times DH$ in a given ratio; and ſo the point D (by lem. 18.) will lie in a conic ſection paſſing thro' the points A, B, C, P, as its locus. *Q. E. D.*

COR. 1. Hence if we draw BC cutting PQ in r, and in PT take Pt to Pr in the ſame ratio which PT has to PR: Then Bt will touch the conic ſection in the point B. For ſuppoſe the point D to coaleſce with the point B, ſo that the chord BD vaniſhing, BT ſhall become a tangent, and CD and BT will coincide with CB and Bt.

COR. 2. And vice verſa, if Bt is a tangent, and the lines BD, CD meet in any point D of a conic ſection; PR will be to PT as Pr to Pt. And on the contrary, if PR is to PT as Pr to Pt, then BD, and CD will meet in ſome point D of a conic ſection.

COR. 3. One conic ſection cannot cut another conic ſection in more than four points. For, if it is poſſible, let two conic ſections paſs thro' the five points A, B, C, P, O; and let the right line BD cut them in the points D, d, and the right line Cd cut the right line PQ in q. Therefore PR is to PT as Pq to PT:

Whence *PR* and *Pq* are equal one to the other, againſt the ſuppoſition.

Lemma XXI.

If two moveable and indeſinite right lines BM, CM *drawn through given points* B, C, *as poles, do by their point of concourſe* M *deſcribe a third right line* MN *given by poſition; and other two indeſinite right lines* BD, CD *are drawn, making with the former two at thoſe given points* B, C, *given angles,* MBD, MCD: *I ſay that thoſe two right lines* BD, CD *will by their point of concourſe* D *deſcribe a conic ſection paſſing through the points* B, C. *And* vice verſa, *if the right lines* BD, CD *do by their point of concourſe* D *deſcribe a conic ſection paſſing through the given points* B, C, A, *and the angle* DBM *is always equal to the given angle* ABC, *as well as the angle* DCM *always equal to the given angle* ACB: *the point* M *will lie in a right line given by poſition, as its locus.* Pl. 9. Fig. 3.

For in the right line *MN* let a point *N* be given, and when the moveable point *M* ſalls on the immoveable point *N*, let the moveable point *D* fall on an immoveable point *P*. Join *CN*, *BN*, *CP*, *BP*, and from the point *P* draw the right lines *PT*, *PR* meeting *BD*, *CD* in *T* and *R*, and making the angle *BPT* equal to the given angle *BNM*, and the angle *CPR* equal to the given angle *CNM*. Wherefore ſince (by ſuppoſition) the angles *MBD*, *NBP* are equal, as alſo the angles *MCD*, *NCP*; take away the angles *NBD* and *NCD* that are common, and there will remain the angles *NBM* and *PBT*, *NCM* and *PCR* equal; and therefore the triangles *NBM*, *PBT* are ſimilar, as alſo the triangles *NCM*, *PCR*. Wherefore *PT* is to *NM*, as *PB* to *NB*; and *PR* to *NM*, as *PC* to *NC*. But the points *B*, *C*, *N*, *P* are immoveable: Wherefore *PT* and *PR* have a given ratio to *NM*, and conſequently a given ratio between themſelves; and therefore, (by lem. 20.) the point *D* wherein the moveable right lines *BT* and *CR* perpetually concur, will be placed in a conic ſection paſſing through the points *B*, *C*, *P*. *Q. E. D.*

And *vice verſa*, if the moveable point *D* (*Pl.* 9. *Fig.* 4.) lies in a conic ſection paſſing through the given points *B*, *C*, *A*; and the angle *DBM* is always equal to the given angle *ABC*, and the angle *DCM* always equal to the given angle *ACB*, and when the point *D* falls ſucceſſively on any two immoveable points *p*, *P*, of the conic ſection, the moveable point *M* falls ſucceſſively on two immoveable points *n*, *N*: through theſe points *n*, *N*, draw the right line *nN*, this line *nN* will be the perpetual locus of that moveable point *M*. For if poſſible, let the point *M* be placed in any curve line. Therefore the point *D* will be placed in a conic ſection paſſing through the five points *B*, *C*, *A*, *p*, *P*, when the point *M* is perpetually placed in a curve line. But from what was demonſtrated before, the point *D* will be alſo placed in a conic ſection, paſſing through the ſame five points *B*, *C*, *A*, *p*, *P*, when the point *M* is perpetually placed in a right line. Wherefore the two conic ſections will both paſs through the ſame five points, againſt corol. 3. lem. 20. It is therefore abſurd to ſuppoſe that the point *M* is placed in a curve line. *Q. E. D.*

PROPOSITION XXII. PROBLEM XIV.

To deſcribe a trajectory that ſhall paſs through five given points. Pl. 9. Fig. 5.

Let the five given points be *A*, *B*, *C*, *P*, *D*. From any one of them as *A*, to any other two as *B*, *C*, which may be called the poles, draw the right lines *AB*, *AC*, and parallel to thoſe the lines *TPS*, *PRQ* through the fourth point *P*. Then from the two poles *B*, *C*, draw through the fifth point *D* two indefinite lines *BDT*, *CRD*, meeting with the laſt drawn lines *TPS*, *PRQ* (the former with the former, and the latter with the latter) in *T* and *R*. Then drawing the right line *tr* parallel to *TR*, cutting off from the right lines *PT*, *PR*, any ſegments *Pt*, *Pr*, proportional to *PT*, *PR*; and if through their extremities *t*, *r*, and the poles *B*, *C*, the right lines *Bt*, *Cr* are drawn, meeting in *d*, that point *d* will be placed in the trajectory required. For (by lem. 20.) that point *d* is placed in a conic ſection paſſing through the four points *A*, *B*, *C*, *P*; and the lines *Rr*, *Tt* vaniſhing, the point *d* comes to coincide with the point *D*. Wherefore the conic ſection paſſes through the five points *A*, *B*, *C*, *P*, *D*. *Q. E. D.*

The ſame otherwiſe. Pl. 9. Fig. 6.

Of the given points join any three as *A, B, C*; and about two of them *B, C*, as poles, making the angles *ABC, ACB* of a given magnitude to revolve, apply the legs *BA, CA*, firſt to the point *D*, then to the point *P*, and mark the points *M, N*, in which the other legs *BL, CL* interſect each the other in both caſes. Draw the indefinite right line *MN*, and let thoſe moveable angles revolve about their poles *B, C*, in ſuch manner that the interſection, which is now ſuppoſed to be *m*, of the legs *BL, CL*, or *BM, CM*, may always fall in that indefinite right line *MN*; and the interſection which is now ſuppoſed to be *d*, of the legs *BA, CA*, or *BD, CD*, will deſcribe the trajectory required *PAD dB*. For, (by lem. 21.) the point *d* will be placed in a conic ſection paſſing through the points *B, C*; and when the point *m* comes to coincide with the points *L, M, N*, the point *d* will (by conſtruction) come to coincide with the points *A, D, P*. Wherefore a conic ſection will be deſcribed that ſhall paſs through the five points *A, B, C, P, D*. *Q. E. F.*

Cor. 1. Hence a right line may be readily drawn which ſhall be a tangent to the trajectory in any given point *B*. Let the point *d* come to coincide with the point *B*, and the right line *Bd* will become the tangent required.

Cor. 2. Hence alſo may be found the centres, diameters, and latera recta of the trajectories, as in cor. 2. lem. 19.

Scholium.

The former of theſe conſtructions (*Fig.* 5.) will become ſomething more ſimple by joining *BP*, and in that line, produced if need be, taking *Bp* to *BP* as *PR* is to *PT*; and through *p* drawing the indefinite right line *pe* parallel to *SPT*; and in that line *pe* taking always *pe* equal to *Pr*; and drawing the right lines *Be*, *Cr* to meet in *d*. For ſince *Pr* to *Pt*, *PR* to *PT*, *pB* to *PB*, *pe* to *Pt*, are all in the ſame ratio, *pe* and *Pr* will be always equal. After this manner the points of the trajectory are moſt readily found, unleſs you would rather deſcribe the curve mechanically as in the ſecond conſtruction.

Proposition XXIII. Problem XV.

To deſcribe a trajectory that ſhall paſs through four given points, and touch a right line given by poſition. Pl. 10. Fig. 1.

Case 1. Suppoſe that *HB* is the given tangent, *B* the point of contact, and *C*, *D*, *P*, the three other given points. Join *BC*, and drawing *PS* parallel to *BH*, and *PQ* parallel to *BC*, compleat the parallelogram *BSPQ*. Draw *BD* cutting *SP* in *T*, and *CD* cutting *PQ* in *R*. Laſtly, drawing any line *tr* parallel to *TR*, cutting off from *PQ*, *PS*, the ſegments *Pr*, *Pt* proportional to *PR*, *PT* reſpectively; and drawing *Cr*, *Bt*, their point of concourſe *d* will (by lem. 20.) always fall on the trajectory to be deſcribed.

The ſame otherwiſe. Pl. 10. Fig. 2.

Let the angle *CBH* of a given magnitude revolve about the pole *B*, as alſo the rectilinear radius *DC* both ways produced, about the pole *C*. Mark the points *M*, *N*, on which the leg *BC* of the angle cuts that radius when *BH* the other leg thereof meets the ſame radius in the points *P* and *D*. Then drawing the indefinite line *MN*, let that radius *CP* or *CD* and the leg *BC* of the angle perpetually meet in this line; and the point of concourſe of the other leg *BH* with the radius will delineate the trajectory required.

For if in the conſtructions of the preceding problem the point *A* comes to a coincidence with the point *B* the lines *CA* and *CB* will coincide, and the line *AB*, in its laſt ſituation, will become the tangent *BH*; and therefore the conſtructions there ſet down will become the ſame with the conſtructions here deſcribed. Wherefore the concourſe of the leg *BH* with the radius will deſcribe a conic ſection paſſing through the points *C*, *D*, *P*, and touching the line *BH* in the point *B*. *Q. E. F.*

Case 2. Suppoſe the four points *B*, *C*, *D*, *P*, (*Pl.* 10. *Fig.* 3.) given, being ſituated without the tangent *HI*. Join each two by the lines *BD*, *CP*, meeting in *G*, and cutting the tangent in *H* and *I*. Cut the tangent in *A* in ſuch manner that *HA* may be to *IA*, as the rectangle under a mean proportional between *CG* and *GP*, and a mean proportional between *BH* and *HD*, is to a rectangle under a mean proportional between *GD* and *GB*, and a mean proportional between *PI* and *IC*; and *A* will be the point of contact. For if *HX*, a parallel to the right line

Fig. 1. p. 110.

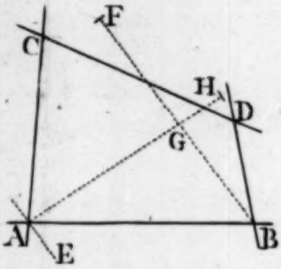


Fig. 2. p. 111.

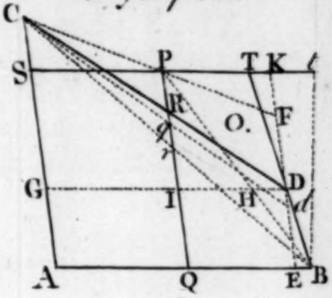


Fig. 3. p. 113.

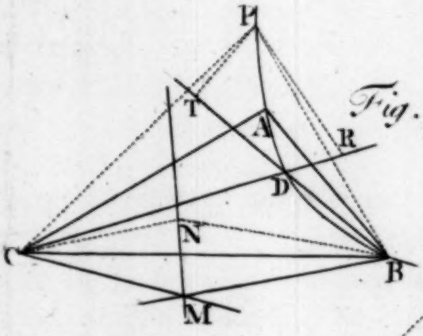


Fig. 4. p. 114.

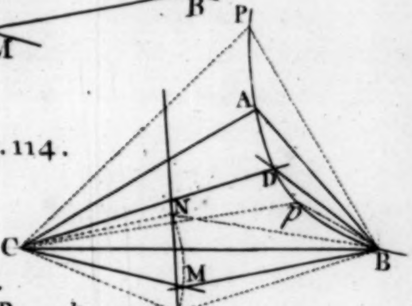


Fig. 5. p. 115. 117.

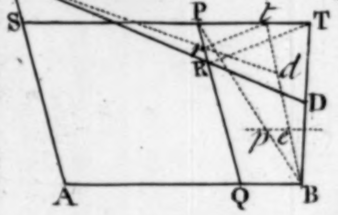
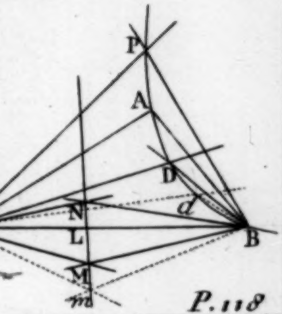


Fig. 6. p. 116.



line *PI*, cuts the trajectory in any points *X* and *Y*, the point *A* (by the properties of the conic ſections) will come to be ſo placed, that HA^2 will become to AI^2 in a ratio that is compounded out of the ratio of the rectangle *XHY* to the rectangle, *BHD*, or of the rectangle *CGP* to the rectangle *DGB*; and the ratio of the rectangle *BHD* to the rectangle *PIC*. But after the point of contact *A* is found, the trajectory will be deſcribed as in the firſt caſe. *Q. E. F.* But the point *A* may be taken either between or without the points *H* and *I*; upon which account a twofold trajectory may be deſcribed.

PROPOSITION XXIV. PROBLEM XVI.

To deſcribe a trajectory that ſhall paſs through three given points, and touch two right lines given by poſition. Pl. 10. Fig. 4.

Suppoſe *HI*, *KL* to be the given tangents, and *B*, *C*, *D*, the given points. Through any two of thoſe points as *B*, *D*, draw the indefinite right line *BD* meeting the tangents in the points *H*, *K*. Then likewiſe through any other two of theſe points as *C*, *D*, draw the indefinite right line *CD*, meeting the tangents in the points *I*, *L*. Cut the lines drawn in *R* and *S*, ſo that *HR* may be to *KR*, as the mean proportional between *BH* and *HD* is to the mean proportional between *BK* and *KD*; and *IS* to *LS*, as the mean proportional between *CI* and *ID* is to the mean proportional between *CL* and *LD*. But you may cut, at pleaſure, either within or between the points *K* and *H*, *I* and *L*, or without them; then draw *RS* cutting the tangents in *A* and *P*, and *A* and *P* will be the points of contact. For if *A* and *P* are ſuppoſed to be the points of contact, ſituated any where elſe in the tangents, and through any of the points *H*, *I*, *K*, *L*, as *I*, ſituated in either tangent *HI*, a right line *IY* is drawn, parallel to the other tangent *KL*, and meeting the curve in *X* and *Y*; and in that right line there be taken *IZ* equal to a mean proportional between *IX* and *IY*; the rectangle *XIY* or IZ^2, will (by the properties of the conic ſections) be to LP^2, as the rectangle *CID* is to the rectangle *CLD*, that is (by the conſtruction) as SI^2 is to SL^2, and therefore *IZ* is to *LP*, as *SI* to *SL*. Wherefore the points *S*, *P*, *Z*, are in one right line. Moreover, ſince the tangents meet in *G*, the rectangle *XIY* or IZ^2 will (by the properties of the conic ſections) be to IA^2 as GP^2 is to GA^2, and conſequently *IZ* will be to *IA*, as *GP* to *GA*. Wherefore the points *P*, *Z*, *A*, lie in one right line, and therefore the points *S*, *P*, and *A* are in one right line. And the ſame argument will prove that the points *R*, *P*, and *A* are in one right line. Wherefore the points of contact *A* and *P* lie in the right line *RS*. But after theſe points are found the trajectory may be deſcribed as in the firſt caſe of the preceding problem. *Q. E. F.*

In this propoſition, and caſe 2. of the foregoing, the conſtructions are the ſame, whether the right line *XY* cut the trajectory in *X* and *Y*, or not; neither do they depend upon that ſection. But the conſtructions being demonſtrated where that right line does cut the trajectory, the conſtructions, where it does not, are alſo known; and therefore, for brevity's ſake, I omit any farther demonſtration of them.

LEMMA XXII.

To transform figures into other figures of the ſame kind. Pl. 10. Fig. 5.

Suppoſe that any figure *HGI* is to be transformed. Draw, at pleaſure, two parallel lines *AO*, *BL*, cutting any third line *AB* given by poſition, in *A* and *B*, and from any point *G* of the figure, draw out any right line *GD*, parallel to *OA*, till it meet the right line *AB*. Then from any given point *O* in the line *OA*, draw to the point *D* the right line *OD*, meeting *BL* in *d*, and from the point of concourſe raiſe the right line *dg* containing any given angle with the right line *BL*, and having ſuch ratio to *Od*, as *DG* has to *OD*; and *g* will be the point in the new figure *hgi*, correſponding to the point *G*. And in like manner the ſeveral points of the firſt figure will give as many correſpondent points of the new figure. If we therefore conceive the point *G* to be carried along by a continual motion through all the points of the firſt figure, the point *g* will be likewiſe carried along by a continual motion through all the points of the new figure, and deſcribe the ſame. For diſtinction's ſake, let us call *DG* the firſt ordinate, *dg* the new ordinate, *AD* the firſt abſciſſa, *ad* the new abſciſſa; *O* the pole, *OD* the abſcinding radius, *OA* the firſt ordinate radius, and *Oa* (by which the parallelogram $OABa$ is compleated) the new ordinate radius.

I ſay, then, that if the point G is placed in a right line given by poſition, the point g will be alſo placed in a right line given by poſition. If the point G is placed in a conic ſection, the point g will be likewiſe placed in a conic ſection. And here I underſtand the circle as one of the conic ſections. But farther, if the point G is placed in a line of the third analytical order, the point g will alſo be placed in a line of the third order, and ſo on in curve lines of higher orders. The two lines in which the points G, g, are placed, will be always of the ſame analytical order. For as ad is to OA, ſo are od to OD, dg to DG, and AB to AD; and therefore AD is equal to $\frac{OA \times AB}{ad}$, and DG equal to $\frac{OA \times dg}{ad}$. Now if the point G is placed in a right line, and therefore, in any equation by which the relation between the abſciſſa AD and the ordinate DG is expreſſed, thoſe indetermined lines AD and DG riſe no higher than to one dimenſion, by writing this equation $\frac{OA \times AB}{ad}$ in place of AD, and $\frac{OA \times dg}{ad}$ in place of DG, a new equation will be produced, in which the new abſciſſa ad and new ordinate dg riſe only to one dimenſion; and which therefore muſt denote a right line. But if AD and DG (or either of them) had riſen to two dimenſions in the firſt equation, ad and dg would likewiſe have riſen to two dimenſions in the ſecond equation. And ſo on in three or more dimenſions. The indetermined lines *ad*, *dg* in the ſecond equation, and *AD*, *DG*, in the firſt will always riſe to the ſame number of dimenſions; and therefore the lines in which the points *G*, *g*, are placed are of the ſame analytical order.

I ſay farther, that if any right line touches the curve line in the firſt figure, the ſame right line transferred the ſame way with the curve into the new figure, will touch that curve line in the new figure, and *vice verſa*. For if any two points of the curve in the firſt figure are ſuppoſed to approach one the other till they come to coincide; the ſame points transferred will approach one the other till they come to coincide in the new figure; and therefore the right lines with which thoſe points are joined will become together tangents of the curves in both figures. I might have given demonſtrations of theſe aſſertions in a more geometrical form; but I ſtudy to be brief.

Whereſore if one rectilinear figure is to be transformed into another we need only transfer the interſections of the right lines of which the firſt figure conſiſts, and through the transferred interſections to draw right lines in the new figure. But if a curvilinear figure is to be transformed we muſt transfer the points, the tangents, and other right lines, by means of which the curve line is defined. This lemma is of uſe in the ſolution of the more difficult problems. For thereby we may transform the propoſed figures if they are intricate into others that are more ſimple. Thus any right lines converging to a point are transformed into parallels; by taking for the firſt ordinate radius any right line that paſſes through the point of concourſe of the converging lines, and that, becauſe their point of concourſe is by this means made to go off in infinitum, and parallel lines are ſuch as tend to a point infinitely remote. And after the problem is ſolved in the new figure, if by the inverſe operations we transform the new into the firſt figure, we ſhall have the ſolution required.

This lemma is alſo of uſe in the ſolution of ſolid problems. For as often as two conic ſections occur by the interſection of which a problem may be ſolved; any one of them may be transformed if it is an hyperbola or a parabola, into an ellipſis, and then this ellipſis may be eaſily changed into a circle. So alſo a right line and a conic ſection in the conſtruction of plane problems, may be transformed into a right line and a circle.

PROPOSITION XXV. PROBLEM XVII.

To deſcribe a trajectory that ſhall paſs through two given points, and touch three right lines given by poſition. Pl. 10. Fig. 6.

Through the concourſe of any two of the tangents one with the other, and the concourſe of the third tangent with the right line which paſſes through the two given points, draw an indefinite right line; and taking this line for the firſt ordinate radius transform the figure by the preceding lemma into a new figure. In this figure thoſe two tangents will become parallel to each other, and the third tangent will be parallel to the right line that paſſes through the two given points. Suppoſe *bi*, *kl* to be thoſe two parallel tangents, *ik* the third tangent, and *hl* a right line parallel thereto, paſſing through thoſe points *a*, *b*, through which the conic ſection ought to paſs in this new figure; and compleating the parallelogram *hikl*, let the right lines *hi*, *ik*, *kl* be ſo cut in *c*, *d*, *e*, that *hc* may be to the ſquare root of the rectangle *ahb*, *ic* to *id*, and *ke* to *kd*, as the ſum of the right lines *hi* and *kl* is to the ſum of the three lines, the firſt whereof is the right line *ik*, and the other two are the ſquare roots of the rectangles *ahb* and *alb*; and *c*, *d*, *e*, will be the points of contact. For by the properties of the conic ſections hc^2 to the rectangle *ahb*, and ic^2 to id^2, and ke^2 to kd^2, and el^2 to the rectangle *alb*, are all in the ſame ratio; and therefore *hc* to the ſquare root of *ahb*, *ic* to *id*, *ke* to *kd*, and *el* to the ſquare root of *alb*, are in the ſubduplicate of that ratio; and by compoſition in the given ratio of the ſum of all the antecedents $hi + kl$, to the ſum of all the conſequents $\sqrt{ahb} + ik + \sqrt{alb}$. Wherefore from that given ratio we have the points of contact *c*, *d*, *e*, in the new figure. By the inverted operations of the laſt lemma, let thoſe points be transferred into the firſt figure, and the trajectory will be there deſcribed by prob. 14. *Q. E. F.* But according as the points *a*, *b*, fall between the points *h*, *l*, or without them, the points *c*, *d*, *e*, muſt be taken either between the points *h*, *i*, *k*, *l*, or without them. If one of the points *a*, *b*, falls between the points *h*, *l*, and the other without the points *h*, *l*, the problem is impoſſible.

PROPOSITION XXVI. PROBLEM. XVIII.

To deſcribe a trajectory that ſhall paſs through a given point, and touch four right lines given by poſition. Pl. 11. Fig. 1.

From the common interſections of any two of the tangents to the common interſection of the other two draw an indefinite right line; and taking this line for the firſt ordinate radius transform the figure (by lem. 22.) into a new figure, and the two pairs of tangents each of which before concurred in the firſt ordinate radius will now become parallel. Let *hi* and *kl*, *ik* and *hl*, be thoſe pairs of parallels compleating the parallelogram *hikl*. And let *p* be the point in this new figure correſponding to the given point in the firſt figure. Through *O* the centre of the figure draw *pq*, and *Oq* being equal to *Op*, *q* will be the other point, through which the conic ſection muſt paſs in this new figure. Let this point be transferred by the inverſe operation of lem. 22. into the firſt figure, and there we ſhall have the two points, through which the trajectory is to be deſcribed. But through thoſe points that trajectory may be deſcribed by prob. 17. *Q.E.F.*

Plate X. Vol. I. P. 126.

Fig. 1. p. 117.

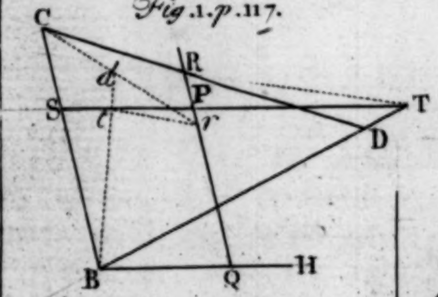


Fig. 2. p. 118.

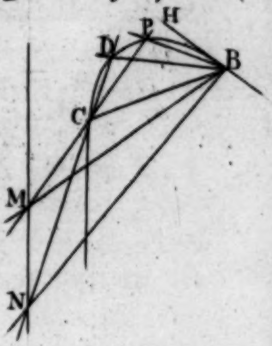


Fig. 3. ibid.

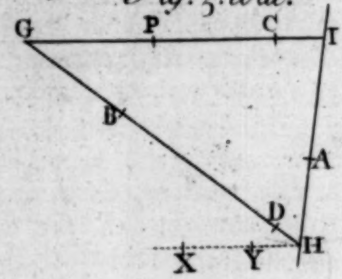


Fig. 4. p. 119.

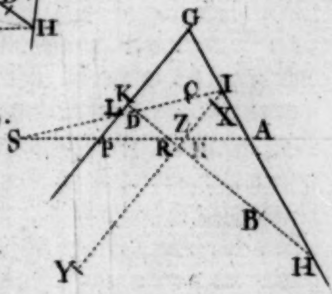
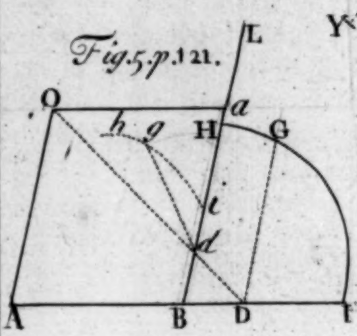


Fig. 5. p. 121.



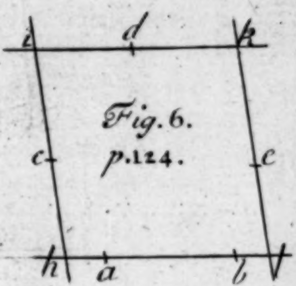


Fig. 6. p. 124.

Lemma XXIII.

If two right lines as AC, BD *given by poſition, and terminating in given points* A, B, *are in a given ratio one to the other, and the right line* CD, *by which the indetermined points* C, D *are joined, is cut in* K *in a given ratio; I ſay that the point* K *will be placed in a right line given by poſition.* Pl. 11. Fig. 2.

For let the right lines *AC*, *BD* meet in *E*, and in *BE* take *BG* to *AE*, as *BD* is to *AC*, and let *FD* be always equal to the given line *EG*; and by conſtruction, *EC* will be to *GD*, that is, to *EF*, as *AC* to *BD*, and therefore in a given ratio; and therefore the triangle *EFC* will be given in kind. Let *CF* be cut in *L* ſo as *CL* may be to *CF* in the ratio of *CK* to *CD*; and becauſe that is a given ratio, the triangle *EFL* will be given in kind, and therefore the point *L* will be placed in the right line *EL* given by poſition. Join *LK* and the triangles *CLK*, *CFD* will be ſimilar; and becauſe *FD* is a given line, and *LK* is to *FD* in a given ratio, *LK* will be alſo given. To this let *EH* be taken equal, and *ELKH* will be always a parallelogram. And therefore the point *K* is always placed in the ſide *HK* (given by poſition) of that parallelogram. *Q. E. D.*

Cor. Becauſe the figure *EFLC* is given in kind, the three right lines *EF*, *EL* and *EC*, that is *GD*, *HK* and *EC* will have given ratio's to each other.

Lemma XXIV.

If three right lines, two whereof are parallel, and given by poſition, touch any conic ſection; I ſay, that the ſemidiameter of the ſection which is parallel to thoſe two is a mean proportional between the ſegments of thoſe two, that are intercepted between the points of contact and the third tangent. Pl. 11. Fig. 3.

Let *AF*, *GB* be the two parallels touching the conic ſection *ADB* in *A* and *B*; *EF* the third right line touching the conic ſection in *I*, and meeting the two former tangents in *F* and *G*, and let *CD* be the ſemi-diameter of the figure parallel to thoſe tangents; I ſay, that *AF*, *CD*, *BG* are continually proportional.

For if the conjugate diameters *AB*, *DM* meet the tangent *FG* in *E* and *H*, and cut one the other in *C*, and the parallelogram *IKCL* be compleated; from the nature of the conic ſections, *EC* will be to *CA* as *CA* to *CL*, and ſo by diviſion, *EC*—*CA* to *CA*—*CL* or *EA* to *AL*; and by compoſition, *EA* to *EA*+*AL* or *EL*, as *EC* to *EC*+*CA* or *EB*; and therefore (becauſe of the ſimilitude of the triangles *EAF*, *ELI*, *ECH*, *EBG*) *AF* is to *LI* as *CH* to *BG*. Likewiſe from the nature of the conic ſections, *LI* (or *CK*) is to *CD* as *CD* to *CH*; and therefore (*ex æquo perturbate*) *AF* is to *CD*, as *CD* to *BG*. *Q. E. D.*

COR. 1. Hence if two tangents *FG*, *PQ* meet two parallel tangents *AF*, *BG* in *F* and *G*, *P* and *Q*, and cut one the other in *O*; *AF* (*ex æquo perturbate*) will be to *BQ*, as *AP* to *BG*, and by diviſion, as *FP* to *GQ*, and therefore as *FO* to *OG*.

COR. 2. Whence alſo the two right lines *PG*, *FQ* drawn through the points *P* and *G*, *F* and *Q* will meet in the right line *ACB* paſſing through the centre of the figure and the points of contact *A*, *B*.

LEMMA XXV.

If four ſides of a parallelogram indefinitely produced touch any conic ſection, and are cut by a fifth tangent; I ſay, that taking thoſe ſegments of any two conterminous ſides that terminate in oppoſite angles of the parallelogram, either ſegment is to the ſide from which it is cut off, as that part of the other conterminous ſide which is intercepted between the point of contact and the third ſide, is to the other ſegment. Pl. 11. Fig. 4.

Let the four ſides *ML*, *IK*, *KL*, *MI* of the parallelogram *MLIK* touch the conic ſection in

A, *B*, *C*, *D*; and let the fifth tangent *FQ* cut thoſe ſides in *F*, *Q*, *H* and *E*, and taking the ſegments *ME*, *KQ* of the ſides *MI*, *KI*; or the ſegments *KH*, *MF* of the ſides *KL*, *ML*; I ſay, that *ME* is to *MI* as *BK* to *KQ*; and *KH* to *KL*, as *AM* to *MF*. For, by cor. 1. of the preceding lemma, *ME* is to *EI*, as (*AM* or) *BK* to *BQ*; and, by compoſition, *ME* is to *MI*, as *BK* to *KQ*. *Q. E. D.* Alſo *KH* is to *HL* as (*BK* or) *AM* to *AF*, and by diviſion *KH* to *KL*, as *AM* to *MF*. *Q. E. D.*

Cor. 1. Hence if a parallelogram *IKLM* deſcribed about a given conic ſection is given, the rectangle $KQ \times ME$, as alſo the rectangle $KH \times MF$ equal thereto, will be given. For, by reaſon of the ſimilar triangles *KQH*, *MFE*, thoſe rectangles are equal.

Cor. 2. And if a ſixth tangent *eq* is drawn meeting the tangents *KI*, *MI* in *q* and *e*; the rectangle $KQ \times ME$ will be equal to the rectangle $Kq \times Me$, and *KQ* will be to *Me*, as *Kq* to *ME*, and by diviſion as *Qq* to *Ee*.

Cor. 3. Hence alſo if *Eq*, *eQ*, are joined and biſected, and a right line is drawn through the points of biſection, this right line will paſs through the centre of the conic ſection. For ſince *Qq* is to *Ee*, as *KQ* to *Me*; the ſame right line will paſs through the middle of all the lines *Eq*, *eQ*, *MK* (by lem. 22.) and the middle point of the right line *MK* is the centre of the ſection.

Proposition XXVII. Problem XIX.

To deſcribe a trajectory that may touch five right lines given by poſition. Pl. 11. Fig. 5.

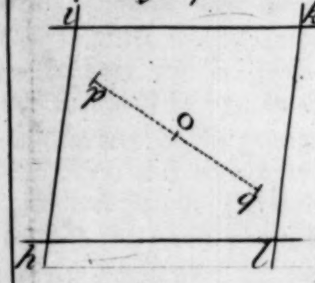
Suppoſing *ABG*, *BCF*, *GCD*, *FDE*, *EA* to be the tangents given by poſition. Biſect in *M* and *N*, *AF*, *BE* the diagonals of the quadrilateral figure *ABFE* contained under any four of them; and (by cor. 3. lem. 25) the right line *MN* drawn through the points of biſection will paſs through the centre of the trajectory. Again, biſect in *P* and *Q* the diagonals (if I may ſo call them) *BD*, *GF* of the quadrilateral figure *BGDF* contained under any other four tangents, and the right line *PQ* drawn through the points of biſection will paſs through the centre of the trajectory. And therefore the centre will be given in the concourſe of the biſecting lines. Suppoſe it to be *O*. Parallel to any tangent *BC* draw *KL*, at ſuch diſtance that the centre *O* may be placed in the middle between the parallels; this *KL* will touch the trajectory to be deſcribed. Let this cut any other two tangents *GCD*, *FDE*, in *L* and *K*. Through the points *C* and *K*, *F* and *L*, where the tangents not parallel *CL*, *FK* meet the parallel tangents *CF*, *KL*, draw *CK*, *FL* meeting in *R*; and the right line *OR* drawn and produced, will cut the parallel tangents *CF*, *KL*, in the points of contact. This appears from cor. 3. lem. 24. And by the ſame method the other points of contact may be found, and then the trajectory may be deſcribed by prob. 14. *Q. E. F.*

SCHOLIUM.

Under the preceding propoſitions are comprehended thoſe problems wherein either the centres or aſymptotes of the trajectories are given. For when points and tangents and the centre are given, as many other points and as many other tangents are given at an equal diſtance on the other ſide of the centre. And an aſymptote is to be conſidered as a tangent, and its infinitely remote extremity (if we may ſay ſo) is a point of contact. Conceive the point of contact of any tangent removed *in infinitum*, and the tangent will degenerate into an aſymptote, and the conſtructions of the preceding problems will be changed into the conſtructions of thoſe problems wherein the aſymptote is given.

After the trajectory is deſcribed, we may find its axes and foci in this manner. In the conſtruction and figure of lem. 21. (*Pl.* 12. *Fig.* 1.) let thoſe legs *BP*, *CP*, of the moveable angles *PBN*, *PCN*, by the concourſe of which the trajectory was deſcribed, be made parallel one to the other; and retaining that poſition, let them revolve about their poles *B*, *C*, in that figure. In the mean while let the other legs *CN*, *BN*, of thoſe angles, by their concourſe *K* or *k* deſcribe the circle *BKGC*. Let *O* be the centre of this circle; and from this centre upon the ruler *MN*, wherein thoſe legs *CN*, *BN* did concur while the trajectory was deſcribed, let fall the perpendicular *OH* meeting the circle in *K* and *L*. And when thoſe other legs *CK*, *BK* meet in the point *K* that is neareſt to the ruler, the firſt legs *CP*, *BP* will be parallel to the greater axis and

Fig. 1. p. 126.



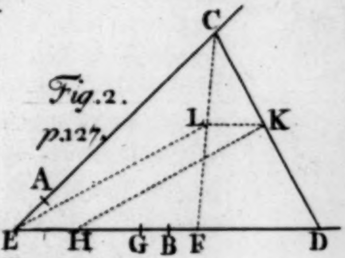
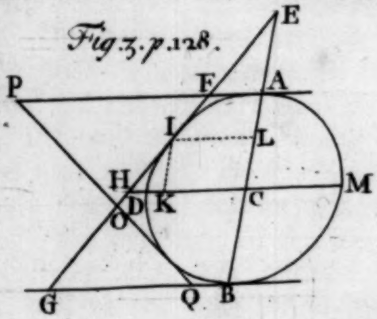


Fig. 2. p. 127.

Fig. 3. p. 128.



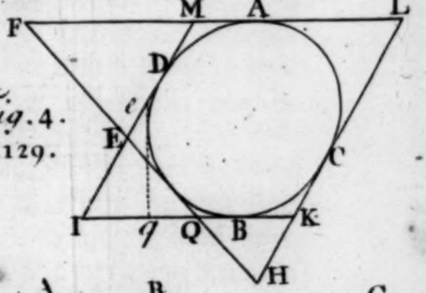


Fig. 4. p. 129.

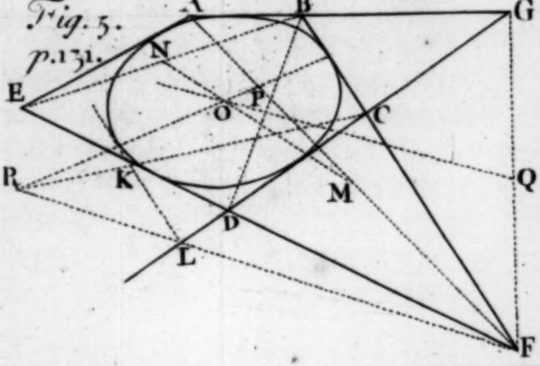


Fig. 5. p. 131.

and perpendicular on the leſſer; and the contrary will happen if thoſe legs meet in the remoteſt point *L*. Whence if the centre of the trajectory is given, the axes will be given; and thoſe being given, the foci will be readily found.

But the ſquares of the axes are one to the other as *KH* to *LH*, and thence it is eaſy to deſcribe a trajectory given in kind through four given points. For if two of the given points are made the poles *C*, *B*, the third will give the moveable angles *PCK*, *PBK*; but thoſe being given, the circle *BGKC* may be deſcribed. Then, becauſe the trajectory is given in kind, the ratio of *OH* to *OK*, and therefore *OH* it ſelf, will be given. About the centre *O*, with the interval *OH*, deſcribe another circle, and the right line that touches this circle and paſſes through the concourſe of the legs *CK*, *BK*, when the firſt legs *CP*, *BP*, meet in the fourth given point, will be the ruler *MN*, by means of which the trajectory may be deſcribed. Whence alſo on the other hand a trapezium given in kind (excepting a few caſes that are impoſſible) may be inſcribed in a given conic ſection.

There are alſo other lemma's by the help of which trajectories given in kind may be deſcribed through given points, and touching given lines. Of ſuch a ſort is this, that if a right line is drawn through any point given by poſition, that may cut a given conic ſection in two points, and the diſtance of the interſections is biſected, the point of biſection will touch another conic ſection of the ſame kind with the former, and having its axes parallel to the axes of the former. But I haſten to things of greater uſe.

LEMMA XXVI.

To place the three angles of a triangle, given both in kind and magnitude, in reſpect of as many right lines given by poſition, provided they are not all parallel among themſelves, in ſuch manner that the ſeveral angles may touch the ſeveral lines. Pl. 12. Fig. 2.

Three indefinite right lines *AB*, *AC*, *BC*, are given by poſition, and it is required ſo to place the triangle *DEF* that its angle *D* may touch the line *AB*, its angle *E* the line *AC*, and its angle *F* the line *BC*. Upon *DE*, *DF* and *EF*, deſcribe three ſegments of circles *DRE*, *DGF*, *EMF*, capable of angles equal to the angles *BAC*, *ABC*, *ACB* reſpectively. But thoſe ſegments are to be deſcribed towards ſuch ſides of the lines *DE*, *DF*, *EF*, that the letters *DRED* may turn round about in the ſame order with the letters *BACB*; the letters *DGFD* in the ſame order with the letters *ABCA*; and the letters *EMFE* in the ſame order with the letters *ACBA*; then compleating thoſe ſegments into entire circles, let the two former circles cut one the other in *G*, and ſuppoſe *P* and *Q* to be their centres. Then joining *GP*, *PQ*, take *Ga* to *AB*, as *GP* is to *PQ*; and about the centre *G*, with the interval *Ga* deſcribe a circle that may cut the firſt circle *DGE* in *a*. Join *aD* cutting the ſecond circle *DFG* in *b*, as well as *aE* cutting the third circle *EMF* in *c*. Compleat the figure *ABCdef* ſimilar and equal to the figure *abcDEF*. I ſay the thing is done.

For drawing *Fc* meeting *aD* in *n*, and joining *aG*, *bG*, *QG*, *QD*, *PD*; by conſtruction the angle *EaD* is equal to the angle *CAB*, and the angle *acF* equal to the angle *ACB*; and therefore the triangle *anc* equiangular to the triangle *ABC*. Wherefore the angle *anc* or *FnD* is equal to the angle *ABC*, and conſequently to the angle *FbD*; and therefore the point *n* falls on the point *b*. Moreover the angle *GPQ* which is half the angle *GPD* at the centre is equal to the angle *GaD* at the circumference; and the angle *GQP*, which is half the angle *GQD* at the centre, is equal to the complement to two right angles of the angle *GbD* at the circumference, and therefore equal to the angle *Gab*. Upon which account the triangles *GPQ*, *Gab*, are ſimilar, and *Ga* is to *ab*, as *GP* to *PQ*; that is (by conſtruction) as *Ga* to *AB*. Wherefore *ab* and *AB* are equal; and conſequently the triangles *abc*, *ABC*, which we have now proved to be ſimilar, are alſo equal. And therefore ſince the angles *D*, *E*, *F*, of the triangle *DEF* do reſpectively touch the ſides *ab*, *ac*, *bc* of the triangle *abc*, the figure *ABCdef* may be compleated ſimilar and equal to the figure *abcDEF*, and by compleating it the problem will be ſolved. *Q. E. F.*

COR. Hence a right line may be drawn whoſe parts given in length may be intercepted between three right lines given by poſition. Suppoſe the triangle *DEF*, by the acceſs of its point *D* to the ſide *EF*, and by having the ſides *DE*, *DF* placed *in directum* to be changed into a right line whoſe given part *DE* is to be interpoſed between the right lines *AB*, *AC* given by poſition; and its given part *DF* is to be interpoſed between the right lines *AB*, *BC*, given by poſition; then by applying the preceding conſtruction to this caſe the problem will be ſolved.

PROPOSITION XXVIII. PROBLEM XX.

To deſcribe a trajectory given both in kind and magnitude, given parts of which ſhall be interpoſed between three right lines given by poſition. Pl. 12. Fig. 3.

Suppoſe a trajectory is to be deſcribed that may be ſimilar and equal to the curve line *DEF*, and may be cut by three right lines *AB*, *AC*, *BC* given by poſition, into parts *DE* and *EF*, ſimilar and equal to the given parts of this curve line.

Draw the right lines *DE*, *EF*, *DF*; and place the angles *D*, *E*, *F*, of this triangle *DEF*, ſo as to touch thoſe right lines given by poſition (by lem. 26.) Then about the triangle deſcribe the trajectory, ſimilar and equal to the curve *DEF*. *Q. E. F.*

Plate XII. Vol I. P. 136.

Fig. 1. pag. 132.

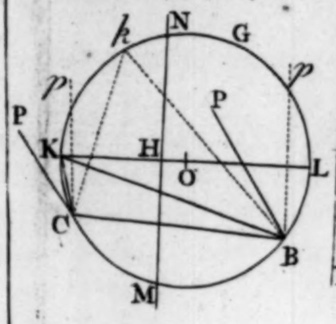


Fig. 1. p. 132.

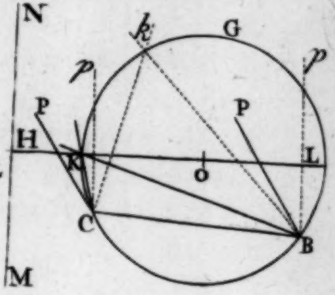


Fig. 2. p. 134.

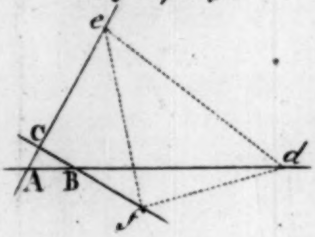


Fig. 2. p. 134.

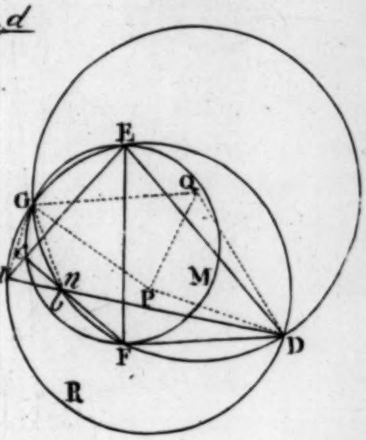
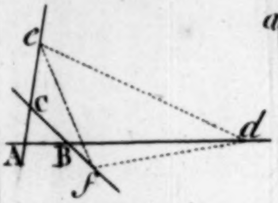


Fig. 3. p. 136.



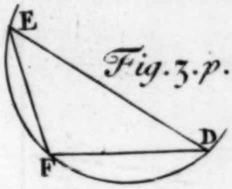


Fig. 3. p. 136.

LEMMA XXVII.

To deſcribe a trapezium given in kind, the angles whereof may be ſo placed in reſpect of four right lines given by poſition, that are neither all parallel among themſelves nor converge to one common point, that the ſeveral angles may touch the ſeveral lines. Pl. 13. Fig. 1.

Let the four right lines *ABC*, *AD*, *BD*, *CE*, be given by poſition; the firſt cutting the ſecond in *A*, the third in *B*, and the fourth in *C*; and ſuppoſe a trapezium *fghi* is to be deſcribed, that may be ſimilar to the trapezium *FGHI*; and whoſe angle *f*, equal to the given angle *F*, may touch the right line *ABC*; and the other angles *g*, *h*, *i*, equal to the other given angles *G*, *H*, *I*, may touch the other lines *AD*, *BD*, *CE*, reſpectively. Join *FH*, and upon *FG*, *FH*, *FI* deſcribe as many ſegments of circles *FSG*, *FTH*, *FVI*; the firſt of which *FSG* may be capable of an angle equal to the angle *BAD*; the ſecond *FTH* capable of an angle equal to the angle *CBD*; and the third *FVI* of an angle equal to the angle *ACE*. But the ſegments are to be deſcribed towards thoſe ſides of the lines *FG*, *FH*, *FI*, that the circular order of the letters *FSGF*, may be the ſame as of the letters *BADB*, and that the letters *FTHF* may turn about in the ſame order as the letters *CBDC*, and the letters *FVIF* in the ſame order as the letters *ACEA*. Compleat the ſegments into entire circles, and let *P* be the centre of the firſt circle *FSG*, *Q* the centre of the ſecond *FTH*. Join and produce both ways the line *PQ*, and in it take *QR* in the ſame ratio to *PQ*, as *BC* has to *AB*. But *QR* is to be taken towards that ſide of the point *Q*, that the order of the letters *P*, *Q*, *R* may be the ſame, as of the letters *A*, *B*, *C*; and about the centre *R* with the interval *RF* deſcribe a fourth circle *FNc* cutting the third circle *FVI* in *c*. Join *Fc* cutting the firſt circle in *a*, and the ſecond in *b*. Draw *aG*, *bH*, *cI*, and let the figure *ABCfghi* be made ſimilar to the figure *abcFGHI*; and the trapezium *fghi* will be that which was required to be deſcribed.

For let the two firſt circles *FSG*, *FTH* cut one the other in *K*; join *PK*, *QK*, *RK*, *aK*, *bK*, *cK*, and produce *QP* to *L*. The angles *FaK*, *FbK*, *FcK* at the circumferences are the halves of the angles *FPK*, *FQK*, *FRK*, at the centres, and therefore equal to *LPK*, *LQK*, *LRK* the halves of thoſe angles. Wherefore the figure *PQRK* is equiangular and ſimilar to the figure *abcK*, and conſequently *ab* is to *bc* as *PQ* to *QR*, that is as *AB* to *BC*. But by conſtruction the angles *fAg*, *fBh*, *fCi* are equal to the angles *FaG*, *FbH*, *FcI*. And therefore the figure *ABCfghi* may be compleated ſimilar to the figure *abcFGHI*. Which done a trapezium *fghi* will be conſtructed ſimilar to the trapezium *FGHI*, and which by its angles *f*, *g*, *h*, *i* will touch the right lines *ABC*, *AD*, *BD*, *CE*. *Q. E. F.*

Cor. Hence a right line may be drawn whoſe parts intercepted in a given order, between four right lines given by poſition, ſhall have a given proportion among themſelves. Let the angles *FGH*, *GHI*, be ſo far increaſed that the right lines *FG*, *GH*, *HI*, may lie *in directum*, and by conſtructing the problem in this caſe, a right line *fghi* will be drawn, whoſe parts *fg*, *gh*, *hi*, intercepted between the four right lines given by poſition, *AB* and *AD*, *AD* and *BD*, *BD* and *CE*, will be one to another as the lines *FG*, *GH*, *HI*, and will obſerve the ſame order among themſelves. But the ſame thing may be more readily done in this manner.

Produce *AB* to *K* (*Pl.* 13. *Fig.* 2.) and *BD* to *L*, ſo as *BK* may be to *AB*, as *HI* to *GH*; and *DL* to *BD* as *GI* to *FG*; and join *KL* meeting the right line *CE* in *i*. Produce *iL* to *M*, ſo as *LM* may be to *iL* as *GH* to *HI*; then draw *MQ* parallel to *LB* and meeting the right line *AD* in *g*, and join *gi* cutting *AB*, *BD* in *f*, *h*. I ſay the thing is done.

For let *Mg* cut the right line *AB* in *Q*, and *AD* the right line *KL* in *S*, and draw *AP* parallel to *BD*, and meeting *iL* in *P*, and *gM* to *Lh* (*gi* to *hi*, *Mi* to *Li*, *GI* to *HI*, *AK* to *BK*) and *AP* to *BL* will be in the ſame ratio. Cut *DL* in *R*, ſo as *DL* to *RL* may be in that ſame ratio; and becauſe *gS* to *gM*, *AS* to *AP*, and *DS* to *DL* are proportional; therefore (*ex æquo*) as *gS* to *Lh*, ſo will *AS* be to *BL*, and *DS* to *RL*; and mixtly *BL*—*RL* to *Lh*—*BL*, as *AS*—*DS* to *gS*—*AS*. That is, *BR* is to *Bh*, as *AD* is to *Ag*, and therefore as *BD* to *gQ*. And alternately *BR* is to *BD*, as *Bh* to *gQ*, or as *fh* to *fg*. But by conſtruction the line *BL* was cut in *D* and *R*, in the ſame ratio as the line *FI* in *G* and *H*; and therefore

BR is to *BD* as *FH* to *FG*. Wherefore *fh* is to *fg*, as *FH* to *FG*. Since therefore *gi* to *hi* likewiſe is as *Mi* to *Li*, that is, as *GI* to *HI*, it is manifeſt that the lines *FI*, *fi*, are ſimilarly cut in *G* and *H*, *g* and *h*. *Q. E. F.*

In the conſtruction of this corollary, after the line *LK* is drawn cutting *CE* in *i*, we may produce *iE* to *V*, ſo as *EV* may be to *Ei*, as *FH* to *HI*, and then draw *Vf* parallel to *BD*. It will come to the ſame, if about the centre *i*, with an interval *IH*, we deſcribe a circle cutting *BD* in *X*, and produce *iX* to *Y*, ſo as *iY* may be equal to *IF*, and then draw *Yf* parallel to *BD*.

Sir *Chriſtopher Wren*, and Dr. *Wallis* have long ago given other ſolutions of this problem.

PROPOSITION XXIX. PROBLEM XXI.

To deſcribe a trajectory given in kind, that may be cut by four right lines given by poſition, into parts given in order, kind, and proportion.

Suppoſe a trajectory is to be deſcribed that may be ſimilar to the curve line *FGHI*, (*Pl.* 13. *Fig.* 3.) and whoſe parts, ſimilar and proportional to the parts *FG*, *GH*, *HI* of the other, may be intercepted between the right lines *AB* and *AD*, *AD* and *BD*, *BD* and *CE* given by poſition, *viz.* the firſt between the firſt pair of thoſe lines, the ſecond between the ſecond, and the third between the third. Draw the right lines *FG*, *GH*, *HI*, *FI*; and (by lem. 27.) deſcribe a trapezium *fghi* that may be ſimilar to the trapezium *FGHI*, and whoſe angles *f*, *g*, *h*, *i*, may touch the right lines given by poſition, *AB*, *AD*, *BD*, *CE*, ſeverally according to their order. And then about this trapezium deſcribe a trajectory, that trajectory will be ſimilar to the curve line *FGHI*.

SCHOLIUM.

This problem may be likewiſe conſtructed in the following manner. Joining *FG*, *GH*, *HI*, *FI*, (*Pl.* 13. *Fig.* 4.) produce *GF* to *V*, and join *FH*, *IG*, and make the angles *CAK*, *DAL* equal to the angles *FGH*, *VFH*. Let *AK*, *AL* meet the right line *BD* in *K* and *L*, and thence draw *KM*, *LN*, of which let *KM* make the angle *AKM* equal to the angle *GHI*, and be it ſelf to *AK*, as *HI* is to *GH*; and let *LN* make the angle *ALN* equal to the angle *FHI*, and be it ſelf to *AL*, as *HI* to *FH*. But *AK*, *KM*, *AL*, *LN* are to be drawn towards thoſe ſides of the lines *AD*, *AK*, *AL*, that the letters *CAKMC*, *ALKA*, *DALND* may be carried round in the ſame order as the letters *FGHIF*; and draw *MN* meeting the right line *CE* in *i*. Make the angle *iEP* equal to the angle *IGF*, and let *PE* be to *Ei*, as *FG* to *GI*; and through *P* draw *PQf* that may with the right line *ADE* contain an angle *PQE* equal to the angle *FIG*, and may meet the right line *AB* in *f*, and join *fi*. But *PE* and *PQ* are to be drawn towards thoſe ſides of the lines *CE*, *PE*, that the circular order of the letters *PEiP* and *PEQP* may be the ſame, as of the letters *FGHIF*, and if upon the line *fi*, in the ſame order of letters, and ſimilar to the trapezium *FGHI*, a trapezium *fghi* is conſtructed, and a trajectory given in kind is circumſcribed about it, the problem will be ſolved.

So far concerning the finding of the orbits. It remains that we determine the motions of bodies in the orbits ſo found.

* * *

Plate XIII. Vol. I. P. 142.

Fig. 1. p. 137.

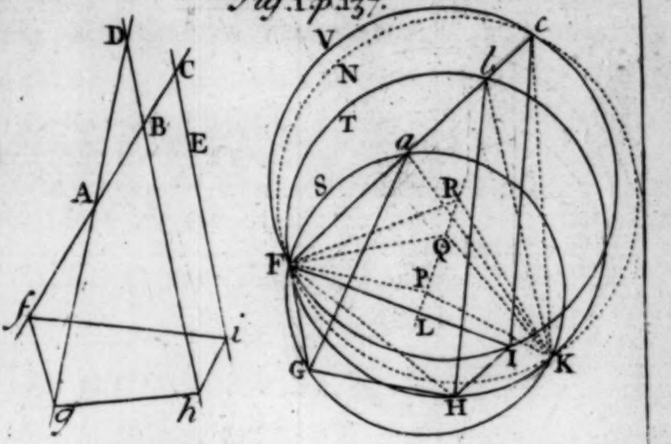


Fig. 2. p. 139.

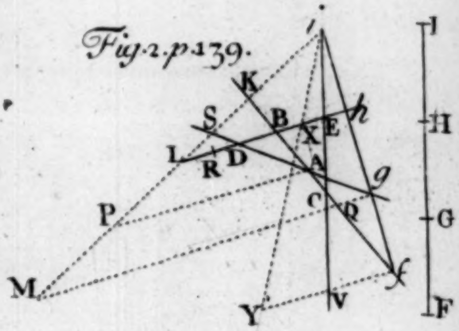
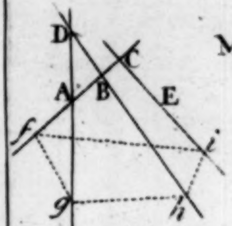


Fig. 3. p. 140.



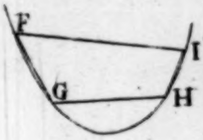
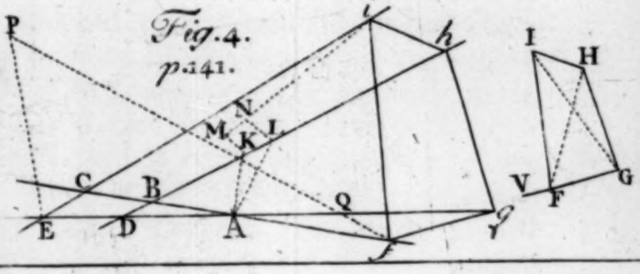


Fig. 3. p. 140.

Fig. 4. p. 141.



SECTION VI.

How the motions are to be found in given orbits.

PROPOSITION XXX. PROBLEM XXII.

To find at any aſſigned time the place of a body moving in a given parabolic trajectory.

Let S (*Pl.* 14. *Fig.* 1.) be the focus, and A the principal vertex of the parabola; and ſuppoſe $4AS \times M$ equal to the parabolic area to be cut off APS, which either was deſcribed by the radius SP, ſince the bodies departure from the vertex, or is to be deſcribed thereby before its arrival there. Now the quantity of that area to be cut off is known from the time which is proportional to it. Biſect AS in G, and erect the perpendicular GH equal to $3M$, and a circle deſcribed about the centre H, with the interval HS, will cut the parabola in the place P required. For letting fall PO

perpendicular on the axis, and drawing PH, there will be $AG^2 + GH^2 \; (= HP^2 = \overline{AO - AG}|^2 + \overline{PO - GH}|^2) = AO^2 + PO^2 - 2GAO - 2GH + PO + AG^2 + GH^2$. Whence $2GH \times PO \; (= AO^2 + PO^2 - 2GAO) = AO^2 + \frac{3}{4}PO^2$. For AO^2 write $AO \times \frac{PO^2}{4AS}$; then dividing all the terms by $3PO$, and multiplying them by $2AS$, we ſhall have $\frac{4}{3}GH \times AS \; (= \frac{1}{6}AO \times PO + \frac{1}{2}AS \times PO = \frac{AO + 3AS}{6} \times PO = \frac{4AO - 3SO}{6} \times PO =$ to the area $\overline{APO - SPO)}| =$ to the area APS. But GH was 3 M, and therefore $\frac{4}{3}GH \times AS$ is $4AS \times$ M. Wherefore the area cut off APS is equal to the area that was to be cut off $4AS \times$ M. *Q. E. D.*

Cor. 1. Hence GH is to AS, as the time in which the body deſcribed the arc AP, to the time in which the body deſcribed the arc between the vertex A and the perpendicular erected from the focus S upon the axis.

Cor. 2. And ſuppoſing a circle ASP, perpetually to paſs through the moving body P, the velocity of the point H, is to the velocity which the body had in the vertex A, as 3 to 8; and therefore in the ſame ratio is the line GH to the right line which the body, in the time of its moving from A to P, would deſcribe with that velocity which it had in the vertex A.

Cor. 3. Hence alſo on the other hand, the time may be found, in which the body has deſcribed any aſſigned arc AP. Join AP, and on its middle point erect a perpendicular meeting the right line GH in H.

LEMMA XXVIII.

There is no oval figure whoſe area, cut off by right lines at pleaſure, can be univerſally found by means of equations of any number of finite terms and dimenſions.

Suppoſe that within the oval any point is given, about which as a pole a right line is perpetually revolving, with an uniform motion, while in that right line a moveable point going out from the pole, moves always forward with a velocity proportional to the ſquare of that right line within the oval. By this motion that point will deſcribe a ſpiral with infinite circumgyrations. Now if a portion of the area of the oval cut off by that right line could be found by a finite equation, the diſtance of the point from the pole, which is proportional to this area, might be found by the ſame equation, and therefore all the points of the ſpiral might be found by a finite equation alſo; and therefore the interſection of a right line given in poſition with the ſpiral might alſo be found by a finite equation. But every right line infinitely produced cuts a ſpiral in an infinite number of points; and the equation by which any one interſection of two lines is found, at the ſame time exhibits all their interſections by as many roots, and thereſore riſes to as many dimenſions as there are interſections. Becauſe two circles mutually cut one another in two points, one of thoſe interſections is not to be found but by an equation of two dimenſions, by which the other interſection may be alſo found. Becauſe there may be four interſections of two conic ſections, any one of them is not to be found univerſally but by an equation of four dimenſions, by which they may be all found together. For if thoſe interſections are ſeverally ſought, becauſe the law and condition of all is the ſame, the calculus will be the ſame in every caſe, and therefore the concluſion always the ſame, which muſt therefore comprehend all thoſe interſections at once within it ſelf, and exhibit them all indifferently. Hence it is that the interſections of the conic ſections with the curves of the third order, becauſe they may amount to ſix, come out together by equations of ſix dimenſions; and the interſections of two curves of the third order, becauſe they may amount to nine come out together by equations of nine dimenſions. If this did not neceſſarily happen, we might reduce all ſolid to plane problems, and thoſe higher than ſolid to ſolid problems. But here I ſpeak of curves irreducible in power. For if the equation by which the curve is defined may be reduced to a lower power, the curve will not be one ſingle curve, but compoſed of two or more; whoſe interſections may be ſeverally found by different calculus's. After the ſame manner the two interſections of right lines with the conic ſections, come out always by equations of two dimenſions; the three interſections of right lines with the irreducible curves of the third order by equations of three dimenſions; the four interſections of right lines with the irreducible curves of the fourth order, by equations of four dimenſions, and ſo on in infinitum. Wherefore the innumerable interſections of a right line with a ſpiral, ſince this is but one ſimple curve, and not reducible to more curves, require equations infinite in number of dimenſions and roots, by which they may be all exhibited together. For the law and calculus of all is the ſame. For if a perpendicular is let fall from the pole upon that interſecting right line, and that perpendicular together with the interſecting line revolves about the pole; the interſections of the ſpiral will mutually paſs the one into the other; and that which was firſt or neareſt, after one revolution, will be the ſecond, after two, the third, and ſo on; nor will the equation in the mean time be changed, but as the magnitudes of thoſe quantities are changed, by which the poſition of the interſecting line is determined. Wherefore ſince thoſe quantities after every revolution return to their firſt magnitudes, the equation will return to its firſt form, and conſequently one and the ſame equation will exhibit all the interſections, and will therefore have an infinite number of roots, by which they may be all exhibited. And therefore the interſection of a right line with a ſpiral cannot be univerſally found by any finite equation; and of conſequence there is no oval figure whoſe area, cut off by right lines at pleaſure, can be univerſally exhibited by any ſuch equation.

By the ſame argument, if the interval of the pole and point by which the ſpiral is deſcribed, is taken proportional to that part of the perimeter of the oval which is cut off; it may be proved that the length of the perimeter cannot be univerſally exhibited by any finite equation. But here I ſpeak of ovals that are not touched by conjugate figures running out in infinitum.

Cor. Hence the area of an ellipſis, deſcribed by a radius drawn from the focus to the moving body, is not to be found from the time given, by a finite equation; and therefore cannot be determined by the deſcription of curves geometrically rational. Thoſe curves I call geometrically rational, all the points whereof may be determined by lengths that are defineable by equations, that is, by the complicated ratio's of lengths. Other curves (ſuch as ſpirals, quadratrixes, and cycloids) I call geometrically irrational. For the lengths which are or are not as number to number (according to the tenth book of elements) are arithmetically rational or irrational. And therefore I cut off an area of an ellipſis proportional to the time in which it is deſcribed by a curve geometrically irrational, in the following manner.

PROPOSITION XXXI. PROBLEM XXIII.

To find the place of a body moving in a given elliptic trajectory at any aſſigned time.

Suppoſe *A* (*Pl.* 14. *Fig.* 2.) to be the principal vertex, *S* the focus, and *O* the centre of the ellipſis *APB*; and let *P* be the place of the body to be found. Produce *OA* to *G*, ſo as *OG* may be to *OA* as *OA* to *OS*. Erect the perpendicular *GH*; and about the centre *O*, with the interval *OG*, deſcribe the circle *GEF*; and on the ruler *GH*, as a baſe, ſuppoſe the wheel *GEF* to move forwards, revolving about its axis, and in the mean time by its point *A* deſcribing the cycloid *ALI*. Which done, take *GK* to the perimeter *GEFG* of the wheel, in the ratio of the time in which the body, proceeding from *A*, deſcribed the arc *AP*, to the time of a whole revolution in the ellipſis. Erect the perpendicular *KL* meeting the cycloid in *L*, then *LP* drawn parallel to *KG* will meet the ellipſis in *P* the required place of the body.

For about the centre *O* with the interval *OA* deſcribe the ſemicircle *AQB*, and let *LP*, produced if need be, meet the arc *AQ* in *Q*, and join *SQ*, *OQ*. Let *OQ* meet the arc *EFG* in *F*, and upon *OQ* let fall the perpendicular *SR*. The area *APS* is as the area *AQS*, that is, as the difference between the ſector *OQA* and the triangle *OQS*, or as the difference of the rectangles $\frac{1}{2}OQ \times AQ$, and $\frac{1}{2}OQ \times SR$, that is, becauſe $\frac{1}{2}OQ$ is given, as the difference between the arc *AQ* and the right line *SR*; and therefore (becauſe of the equality of the given ratio's *SR* to the ſine of the arc *AQ*, *OS* to *OA*, *OA* to *OG*, *AQ* to *GF*, and by diviſion, $AQ - SR$ to $GF -$ ſine of the arc *AQ*) as *GK* the difference between the arc *GF* and the ſine of the arc *AQ*. *Q. E. D.*

SCHOLIUM.

But ſince the deſcription of this curve is difficult, a ſolution by approximation will be preferable. Firſt then let there be found a certain angle B which may be to an angle of 57,29578 degrees, which an arc equal to the radius ſubtends, as *SH* (*Pl.* 14. *Fig.* 3.) the diſtance of the foci, to *AB* the diameter of the ellipſis. Secondly, a certain length L, which may be to the radius in the ſame ratio inverſely. And theſe being found, the problem may be ſolved by the following analyſis. By any conſtruction (or even by conjecture) ſuppoſe we know *P* the place of the body near its true place *p*. Then letting fall on the axis of the ellipſis the ordinate *PR*, from the proportion of the diameters of the ellipſis, the ordinate *RQ* of the circumſcribed circle *AQB* will be given; which ordinate is the ſine of the angle *AOQ* ſuppoſing *AO* to be the radius, and alſo cuts the ellipſis in *P*. It will be ſufficient if that angle is found by a rude calculus in numbers near the truth. Suppoſe we alſo know the angle proportional to the time, that is, which is to four right angles, as the time in which the body deſcribed the arc *Ap*, to the time of one revolution in the ellipſis. Let this angle be N. Then take an angle D, which may be to the angle B as the ſine of the angle *AOQ* to the radius; and an angle E which may be to the angle $N - AOQ + D$, as the length L to the ſame length L diminiſhed by the co-ſine of the angle *AOQ*, when that angle is leſs than a right angle, or increaſed thereby when greater. In the next place take an angle F that may be to the angle B, as the ſine of the angle $AOQ + E$ to the radius, and an angle G, that may be to the angle $N - AOQ - E + F$, as the length L to the ſame length L diminiſhed by the co-ſine of the angle $AOQ + E$, when that angle is leſs than a right angle, or increaſed thereby when greater. For the third time take an angle H, that may be to the angle B as the ſine of the angle $AOQ+E+G$ to the radius; and an angle I to the angle $N-AOQ-E-G+H$, as the length L is to the ſame length L diminiſhed by the co-ſine of the angle $AOQ+E+G$ when that angle is leſs than a right angle, or increaſed thereby when greater. And ſo we may proceed in infinitum. Laſtly, take the angle AOq equal to the angle $AOQ+E+G+I+$ *&c.* and from its co-ſine Or and the ordinate pr, which is to its ſine qr as the leſſer axis of the ellipſis to the greater, we ſhall have p the correct place of the body. When the angle $N-AOQ+D$ happens to be negative, the ſign + of the angle E muſt be every where changed into −, and the ſign − into +. And the ſame thing is to be underſtood of the ſigns of the angles G and I, when the angles $N-AOQ-E+F$, and $N-AOQ-E-G+H$ come out negative. But the infinite ſeries $AOQ+E+G+I+$ *&c.* converges ſo very faſt, that it will be ſcarcely ever needful to proceed beyond the ſecond term E. And the calculus is founded upon this theorem, that the area APS is as the difference between the arc AQ and the right line let fall from the focus S perpendicularly upon the radius OQ.

And by a calculus not unlike, the problem is ſolved in the hyperbola. Let its centre be O, (*Pl.* 14. *Fig.* 4.) its vertex A, its focus S, and aſymptote OK. And ſuppoſe the quantity of the area to be cut off is known, as being proportional to the time. Let that be A, and by conjecture ſuppoſe we know the poſition of a right line SP, that cuts off an area APS near the truth. Join OP, and from A and P to the aſymptote draw AI, PK parallel to the other aſymptote:

and by the table of logarithms the area *AIKP* will be given, and equal thereto the area *OPA*, which ſubducted from the triangle *OPS*, will leave the area cut off *APS*. And by applying 2*APS*—2A, or 2A—2*APS*, the double difference of the area *A* that was to be cut off, and the area *APS* that is cut off, to the line *SN* that is let fall from the focus *S*, perpendicular upon the tangent *TT*, we ſhall have the length of the chord *PQ*. Which chord *PQ* is to be inſcribed between *A* and *P*, if the area *APS* that is cut off be greater than the area A that was to be cut off, but towards the contrary ſide of the point *P*, if otherwiſe: and the point *Q* will be the place of the body more accurately. And by repeating the computation the place may be ſound perpetually to greater and greater accuracy.

And by ſuch computations we have a general analytical reſolution of the problem. But the particular calculus that follows, is better fitted ſor aſtronomical purpoſes. Suppoſing *AO*, *OB*, *OD* (*Pl.* 14. *Fig.* 5.) to be the ſemi-axes of the ellipſis, and L its latus rectum, and D the difference betwixt the leſſer ſemi-axis *OD*, and $\frac{1}{2}$L the half of the latus rectum: let an angle Y be ſound, whoſe ſine may be to the radius, as the rectangle under that difference D and *AO*+*OD* the half ſum of the axes, to the ſquare of the greater axis *AB*. Find alſo an angle Z whoſe ſine may be to the radius, as the double rectangle under the diſtance of the foci *SH* and that difference D, to triple the ſquare of half the greater ſemi-axis *AO*. Thoſe angles being once found, the place of the body may be thus determined. Take the angle T proportional to the time in which the arc *BP* was deſcribed, or equal to what is called the mean motion; and an angle V, the firſt equation of the mean motion to the angle Y, the greateſt firſt equation, as the ſine of double the angle T is to the radius; and an angle X, the ſecond equation, to the angle Z, the ſecond greateſt equation, as the cube of the ſine of the angle T is to the cube of the radius. Then take the angle *BHP* the mean motion equated equal to T + X + V the ſum of the angles T, V, X, if the angle T is leſs than a right angle; or equal to T + X—V the difference of the ſame, if that angle T is greater than one and leſs than two right angles; and if *HP* meets the ellipſis in *P*, draw *SP*, and it will cut off the area *BSP* nearly proportional to the time.

This practice ſeems to be expeditious enough, becauſe the angles V and X, taken in ſecond minutes if you pleaſe, being very ſmall, it will be ſufficient to find two or three of their firſt figures. But it is likewiſe ſufficiently accurate to anſwer to the theory of the planets motions. For even in the orbit of Mars, where the greateſt equation of the centre amounts to ten degrees, the error will ſcarcely exceed one ſecond. But when the angle of the mean motion equated *BHP* is found, the angle of the true motion *BSP*, and the diſtance *SP* are readily had by the known methods.

And ſo far concerning the motion of bodies in curve lines. But it may alſo come to paſs that a moving body ſhall aſcend or deſcend in a right line; and I ſhall now go on to explain what belongs to ſuch kind of motions.

SECTION VII.

Concerning the rectilinear aſcent and deſcent of bodies.

PROPOSITION XXXII. PROBLEM XXIV.

Suppoſing that the centripetal force is reciprocally proportional to the ſquare of the diſtance of the places from the centre; it is required to define the ſpaces which a body, falling directly, deſcribes in given times.

CASE 1. If the body does not fall perpendicularly it will (by cor. 1. prop. 13.) deſcribe ſome conic ſection whoſe focus is placed in the centre of force. Suppoſe that conic ſection to be *ARPB* (*Pl.* 15. *Fig.* 1.) and its focus *S*. And firſt, if the figure be an ellipſis; upon the greater axe thereof *AB* deſcribe the ſemi-circle *ADB*, and let the right line *DPC* paſs through the falling body, making

Plate XIV. Vol. I. P. 154.

Fig. 2. p. 148.

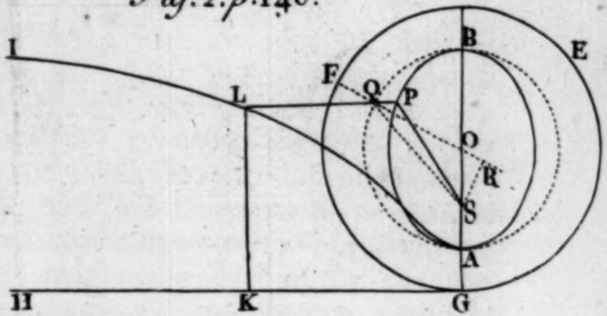


Fig. 1. p. 143.

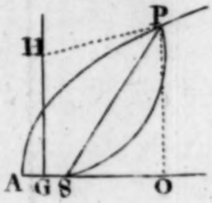


Fig. 3. p. 150.

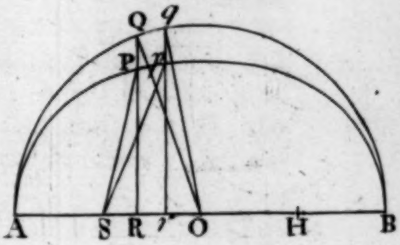


Fig. 5. p. 152.

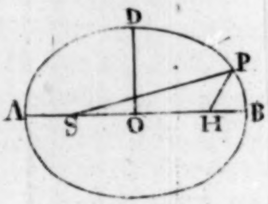
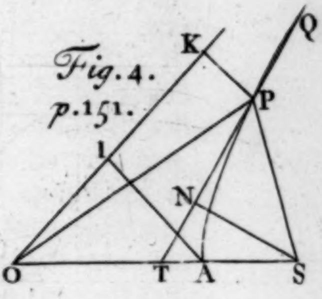


Fig. 4. p. 151.



making right angles with the axis; and drawing *DS*, *PS*, the area *ASD* will be proportional to the area *ASP*, and therefore alſo to the time. The axis *AB* ſtill remaining the ſame, let the breadth of the ellipſis be perpetually diminiſhed, and the area *ASD* will always remain proportional to the time. Suppoſe that breadth to be diminiſhed in infinitum; and the orbit *APB* in that caſe coinciding with the axis *AB*, and the focus *S* with the extreme point of the axis *B*, the body will deſcend in the right line *AC*, and the area *ABD* will become proportional to the time. Wherefore the ſpace *AC* will be given which the body deſcribes in a given time by its perpendicular fall from the place *A*, if the area *ABD* is taken proportional to the time, and from the point *D*, the right line *DC* is let fall perpendicularly on the right line *AB*. *Q. E. I.*

CASE 2. If the figure *RPB* is an hyperbola, (*Fig.* 2.) on the ſame principal diameter *AB* deſcribe the rectangular hyperbola *BED*; and becauſe the areas *CSP*, *CBfP*, *SPfB*, are ſeverally to the ſeveral areas *CSD*, *CBED*, *SDEB* in the given ratio of the heights *CP*, *CD*; and the area *SPfB* is proportional to the time in which the body *P* will move through the arc *PfB*, the area *SDEB* will be alſo proportional to that time. Let the latus rectum of the hyperbola *RPB* be diminiſhed in infinitum, the latus tranſverſum remaining the ſame; and the arc *PB* will come to coincide with the right line *CB*, and the focus *S* with the vertex *B*, and the right line *SD* with the right line *BD*. And therefore the area *BDEB* will be proportional to the time in which the body *C*, by its perpendicular deſcent, deſcribes the line *CB*. *Q. E. I.*

Case 3. And by the like argument if the figure *RPB* is a parabola, (*Fig.* 3.) and to the ſame principal vertex *B* another parabola *BED* is deſcribed, that may always remain given while the former parabola in whoſe perimeter the body *P* moves, by having its latus rectum diminiſhed and reduced to nothing, comes to coincide with the line *CB*; the parabolic ſegment *BDEB* will be proportional to the time, in which that body *P* or *C* will deſcend to the centre *S* or *B*. *Q. E. I.*

Proposition XXXIII. Theorem IX.

The things above found being ſuppoſed, I ſay that the velocity of a falling body in any place C, *is to the velocity of a body, deſcribing a circle about the centre* B *at the diſtance* BC, *in the ſubduplicate ratio of* AC, *the diſtance of the body from the remoter vertex* A *of the circle or rectangular hyperbola, to* ½ AB *the principal ſemi-diameter of the figure.* Pl. 15. Fig. 4.

Let *AB* the common diameter of both figures *RPB*, *DEB* be biſected in *O*; and draw the right line *PT* that may touch the figure *RPB* in *P*, and likewiſe cut that common diameter *AB* (produced if need be) in *T*; and let *ST* be perpendicular to this line, and *BQ* to this diameter, and ſuppoſe the latus rectum of the figure RPB to be L. From cor. 9. prop. 16. it is manifeſt that the velocity of a body, moving in the line RPB about the centre S, in any place P, is to the velocity of a body deſcribing a circle about the ſame centre, at the diſtance SP, in the ſubduplicate ratio of the rectangle $\frac{1}{2}L \times SP$ to SY^2. For by the properties of the conic ſections ACB is to CP^2 as $2AO$ to L, and therefore $\frac{2CP^2 \times AO}{ACB}$ is equal to L. Therefore thoſe velocities are to each other in the ſubduplicate ratio of $\frac{CP^2 \times AO \times SP}{ACB}$ to SY^2. Moreover by the properties of the conic ſections, CO is to BO as BO to TO, and (by compoſition or diviſion) as CB to BT. Whence (by diviſion or compoſition) $BO -$ or $+ CO$ will be to BO, as CT to BT, that is, AC will be to AO as CP to BQ; and therefore $\frac{CP^2 \times AO \times SP}{ACB}$ is equal to $\frac{BQ^2 \times AC \times SP}{AO \times BC}$. Now ſuppoſe CP the breadth of the figure RPB to be diminiſhed in infinitum, ſo as the point P may come to coincide with the point C, and the point S with the point B, and the line SP with the line BC, and the line SY with the line BQ; and the velocity of the body now deſcending perpendicularly in the line CB will be to the velocity of a body deſcribing a circle about the centre B at the diſtance BC, in the ſubduplicate ratio of $\frac{BQ^2 \times AC \times SP}{AO \times BC}$ to SY^2, that is (neglecting the ratio's of equality of

SP to *BC*, and BQ^2 to SY^2) in the ſubduplicate ratio of *AC* to *AO* or $\frac{1}{2}AB$. *Q. E. D.*

Cor. 1. When the points *B* and *S* come to coincide, *TC* will become to *TS*, as *AC* to *AO*.

Cor. 2. A body revolving in any circle at a given diſtance from the centre, by its motion converted upwards will aſcend to double its diſtance from the centre.

Proposition XXXIV. Theorem X.

If the figure BED *is a parabola, I ſay that the velocity of a falling body in any place* C *is equal to the velocity by which a body may uniformly deſcribe a circle about the centre* B *at half the interval* BC. Pl. 15. Fig. 5.

For (by cor. 7. prop. 16.) the velocity of a body deſcribing a parabola *RPB* about the centre *S*, in any place *P*, is equal to the velocity of a body uniformly deſcribing a circle about the ſame centre *S* at half the interval *SP*. Let the breadth *CP* of the parabola be diminiſhed in infinitum, ſo as the parabolic arc *PfB* may come to coincide with the right line *CB*, the centre *S* with the vertex *B*, and the interval *SP* with the interval *BC*, and the propoſition will be manifeſt. *Q. E. D.*

Plate XV. Vol. I. P. 158.

Fig. 1. p. 154.

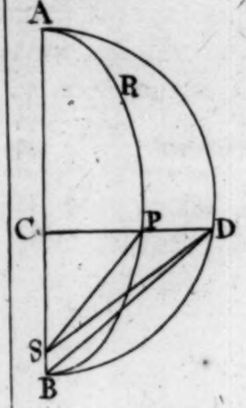


Fig. 2. p. 155.

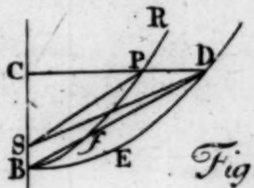


Fig. 3. p. 156.

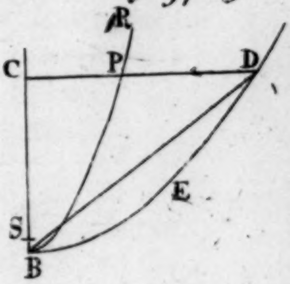


Fig. 4. p. 156.

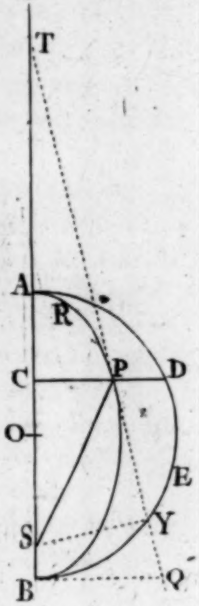


Fig. 4. p. 156.

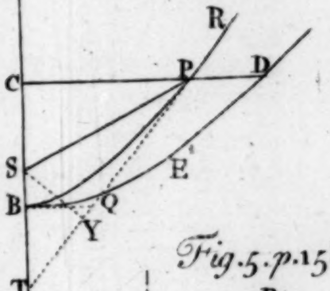
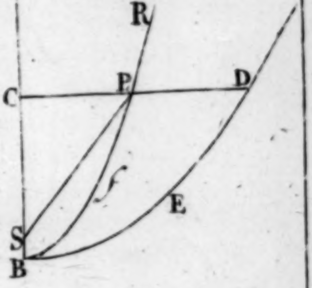


Fig. 5. p. 158.



Proposition XXXV. Theorem XI.

The ſame things ſuppoſed, I ſay that the area of the figure DES, *deſcribed by the indefinite radius* SD, *is equal to the area which a body with a radius equal to half the latus rectum of the figure* DES, *by uniformly revolving about the centre* S, *may deſcribe in the ſame time.* Pl. 16. Fig. 1.

For ſuppoſe a body C in the ſmalleſt moment of time deſcribes in falling the infinitely little line Cc, while another body K uniformly revolving about the centre S in the circle OKk, deſcribes the arc Kk. Erect the perpendiculars CD, cd, meeting the figure DES in D, d. Join SD, Sd, SK, Sk, and draw Dd meeting the axis AS in T, and thereon let fall the perpendicular ST.

Case 1. If the figure DES is a circle or a rectangular hyperbola, biſect its tranſverſe diameter AS in O, and SO will be half the latus rectum. And becauſe TC is to TD as Cc to Dd, and TD to TS as CD to ST; *ex æquo* TC will be to TS, as $CD \times Cc$ to $ST \times Dd$. But (by cor. 1. prop. 33) TC is to TS as AC to AO, to wit, if in the coaleſcence of the points D, d, the ultimate ratio's of the lines are taken. Wherefore AC is to AO or SK as $CD \times Cc$ to $ST \times Dd$. Farther, the velocity of the deſcending body in C is to the velocity of a body deſcribing a circle about the centre S, at the interval SC, in the ſubduplicate ratio of AC to AO or SK (by prop. 33.) and this velocity is to the velocity of a body deſcribing the circle OKk in the ſubduplicate ratio of SK to SC (by cor. 6. prop. 4.) and *ex æquo*, the firſt velocity to the laſt, that is the little line Cc to the arc Kk, in the ſubduplicate ratio of AC to SC, that is in the ratio of AC to CD. Wherefore $CD \times Cc$ is equal to $AC \times Kk$, and conſequently AC to SK as $AC \times Kk$ to $SY \times Dd$, and thence $SK \times Kk$ equal to $SY \times Dd$, and $\frac{1}{2} SK \times Kk$ equal to $\frac{1}{2} SY \times Dd$, that is, the area KSk equal to the area SDd. Therefore in every moment of time two equal particles, KSk and SDd, of areas are generated which, if their magnitude is diminiſhed and their number increaſed in infinitum, obtain the ratio of equality, and conſequently (by cor. lem. 4.) the whole areas together generated are always equal. *Q. E. D.*

CASE 2. But if the figure DES (*Fig.* 2.) is a parabola, we ſhall find as above $CD \times Cc$ to $SY \times Dd$ as TC to TS, that is, as 2 to 1; and that therefore $\frac{1}{4} CD \times Cc$ is equal to $\frac{1}{2} SY \times Dd$. But the velocity of the falling body in C is equal to the velocity with which a circle may be uniformly deſcribed at the interval $\frac{1}{2} SC$, (by prop. 34.) And this velocity to the velocity with which a circle may be deſcribed with the radius SK; that is, the little line Cc to the arc Kk, is (by cor. 5. prop. 4.) in the ſubduplicate ratio of SK to $\frac{1}{2} SC$; that is, in the ratio of SK to $\frac{1}{2} CD$. Wherefore $\frac{1}{2} SK \times Kk$ is equal to $\frac{1}{4} CD \times Cc$, and therefore equal to $\frac{1}{2} SY \times Dd$; that is, the area KSk is equal to the area SDd as above. *Q. E. D.*

Proposition XXXVI. Problem XXV.

To determine the times of the deſcent of a body falling from a given place A. Pl. 16. Fig. 3.

Upon the diameter *AS*, the diſtance of the body from the centre at the beginning, deſcribe the ſemi-circle *ADS*, as likewiſe the ſemi-circle *OKH* equal thereto, about the centre *S*. From any place *C* of the body, erect the ordinate *CD*. Join *SD*, and make the ſector *OSK* equal to the area *ASD*. It is evident by prop. 35. that the body in falling will deſcribe the ſpace *AC* in the ſame time in which another body, uniformly revolving about the centre *S*, may deſcribe the arc *OK*. *Q. E. F.*

Proposition XXXVII. Problem XXVI.

To define the times of the aſcent or deſcent of a body projected upwards or downwards from a given place. Pl. 16. Fig. 4.

Suppoſe the body to go off from the given place *G*, in the direction of the line *GS*, with any velocity. In the duplicate ratio of this velocity to the uniform velocity in a circle, with which the body may revolve about the centre

tre *S* at the given interval *SG*, take *GA* to $\frac{1}{2}AS$. If that ratio is the ſame as of the number 2 to 1 the point *A* is infinitely remote; in which caſe a parabola is to be deſcribed with any latus rectum to the vertex *S*, and axis *SG*; as appears by prop. 34. But if that ratio is leſs or greater than the ratio of 2 to 1, in the former caſe a circle, in the latter a rectangular hyperbola, is to be deſcribed on the diameter *SA*; as appears by prop. 33. Then about the centre *S*, with an interval equal to half the latus rectum, deſcribe the circle *HkK*, and at the place *G* of the aſcending or deſcending body, and at any other place *C*, erect the perpendiculars *GI*, *CD*; meeting the conic ſection or circle in *I* and *D*. Then joining *SI*, *SD*, let the ſectors *HSK*, *HSk* be made equal to the ſegments *SEIS*, *SEDS*, and by prop. 35. the body *G* will deſcribe the ſpace *GC* in the ſame time in which the body *K* may deſcribe the arc *Kk*. *Q. E. F.*

PROPOSITION XXXVIII. THEOREM XII.

Suppoſing that the centripetal force is proportional to the altitude or diſtance of places from the centre, I ſay, that the times and velocities of falling bodies, and the ſpaces which they deſcribe, are reſpectively proportional to the arcs, and the right and verſed ſines of the arcs. Pl. 17. Fig. 1.

Suppoſe the body to fall from any place *A* in the right line *AS*; and about the centre of force *S* with

Plate XVI. Vol. I. P. 162.

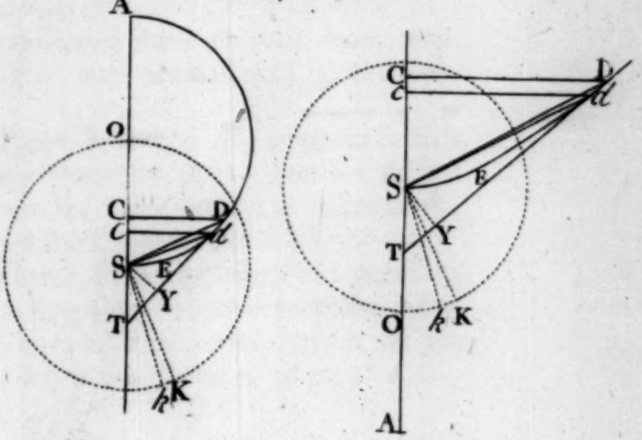


Fig. 1. p. 159.

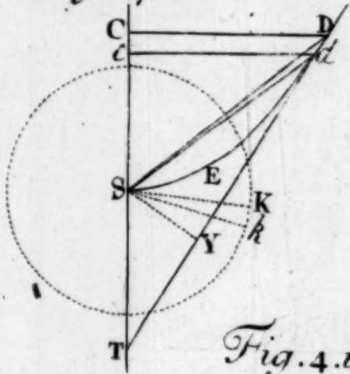


Fig. 2. p. 160.

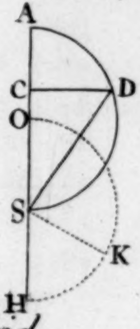


Fig. 3. p. 161.

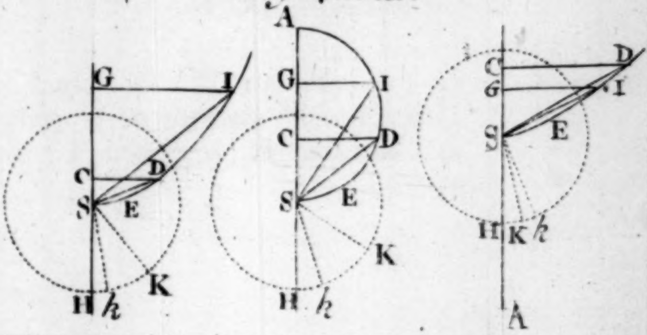


Fig. 4. ibid.

with the interval *AS*, deſcribe the quadrant of a circle *AE*; and let *CD* be the right ſine of any arc *AD*; and the body *A* will in the time *AD* in falling deſcribe the ſpace *AC*, and in the place *C* will acquire the velocity *CD*.

This is demonſtrated the ſame way from prop. 10. as prop. 32. was demonſtrated from prop. 11.

COR. 1. Hence the times are equal in which one body falling from the place *A* arrives at the centre *S*, and another body revolving deſcribes the quadrantal arc *ADE*.

COR. 2. Wherefore all the times are equal in which bodies falling from whatſoever places arrive at the centre. For all the periodic times of revolving bodies are equal, by cor. 3. prop. 4.

PROPOSITION XXXIX. PROBLEM XXVII.

Suppoſing a centripetal force of any kind, and granting the quadratures of curvilinear figures; it is required to find the velocity of a body, aſcending or deſcending in a right line, in the ſeveral places through which it paſſes; as alſo the time in which it will arrive at any place; And vice verſa.

Suppoſe the body *E* (*Pl.* 17. *Fig.* 2.) to fall from any place *A* in the right line *ADEC*; and from its place *E* imagine a perpendicular *EG* always erected, proportional to the centripetal force in that place tending to the centre *C*; and let *BFG* be a curve line, the locus of the point *G*. And in the beginning of the motion ſuppoſe *EG* to coincide with the perpendicular *AB*; and the velocity of the body in any place *E* will be as a right line whoſe power is the curvilinear area *ABGE*. *Q. E. I.*

In *EG* take *EM* reciprocally proportional to a right line whoſe power is the area *ABGE*, and let *VLM* be a curve line wherein the point *M* is always placed, and to which the right line *AB* produced is an aſymptote, and the time in which the body in falling deſcribes the line *AE*, will be as the curvilinear area *ABTVME*. *Q. E. I.*

For in the right line *AE* let there be taken the very ſmall line *DE* of a given length, and let *DLF* be the place of the line *EMG*, when the body was in *D*; and if the centripetal force be ſuch, that a right line whoſe power is the area *ABGE*, is as the velocity of the deſcending body, the area it ſelf will be as the ſquare of that velocity; that is, if for the velocities in *D* and *E* we write V and V+I, the area *ABFD* will be as VV, and the area *ABGE* as VV+2VI+II; and by diviſion the area *DFGE* as 2VI+II, and therefore $\frac{DFGE}{DE}$ will be as $\frac{2VI+II}{DE}$; that is, if we take the firſt ratio's of thoſe quantities when juſt naſcent, the length *DF* is as the quantity $\frac{2VI,}{DE}$ and therefore alſo as half that quantity $\frac{I\times V}{DE}$. But the time, in which the body in falling deſcribes the very ſmall line *DE* is as that line directly and the velocity V inverſely, and the force will be as the increment I of the velocity directly and the time inverſely, and therefore if we take the firſt ratio's when thoſe quantities are juſt naſcent as $\frac{I \times V}{DE}$, that is as the length *DF*. Therefore a force proportional to *DF* or *EG* will cauſe the body to deſcend with a velocity that is as the right line whoſe power is the area *ABGE*, *Q. E. D.*

Moreover ſince the time, in which a very ſmall line *DE* of a given length may be deſcribed, is as the velocity inverſely, and therefore alſo inverſely as a right line whoſe ſquare is equal to the area *ABFD*; and ſince the line *DL*, and by conſequence the naſcent area *DLME*, will be as the ſame right line inverſely: the time will be as the area *DLME*, and the ſum of all the times will be as the ſum of all the area's; that is (by cor. lem. 4.) the whole time in which the line *AE* is deſcribed will be as the whole area *ATVME*. *Q. E. D.*

COR. 1. Let *P* be the place from whence a body ought to fall, ſo as that when urged by any known uniform centripetal force (ſuch as gravity is vulgarly ſuppoſed to be) it may acquire in the place *D* a velocity, equal to the velocity which another body, falling by any force whatever, hath acquired in that place *D*. In the perpendicular *DF* let there be taken *DR*, which may be to *DF* as that uniform force to the other force in the place *D*. Compleat the rectangle *PDRQ*, and cut off the area *ABFD* equal

to

to that rectangle. Then *A* will be the place from whence the other body fell. For compleating the rectangle *DRSE*, ſince the area *ABFD* is to the area *DFGE* as VV to 2VI, and therefore as $\frac{1}{2}$V to I, that is, as half the whole velocity to the increment of the velocity of the body falling by the unequable force; and in like manner the area *PQRD* to the area *DRSE*, as half the whole velocity to the increment of the velocity of the body falling by the uniform force; and ſince thoſe increments (by reaſon of the equality of the naſcent times) are as the generating forces, that is, as the ordinates *DF*, *DR*, and conſequently as the naſcent area's *DFGE*, *DRSE*; therefore *ex æquo* the whole areas *ABFD*, *PQRD* will be to one another as the halves of the whole velocities, and therefore, becauſe the velocities are equal, they become equal alſo.

COR. 2. Whence if any body be projected either upwards or downwards with a given velocity from any place *D*, and there be given the law of centripetal force acting on it, its velocity will be found in any other place as *e*, by erecting the ordinate *eg*, and taking that velocity to the velocity in the place *D*, as a right line whoſe power is the rectangle *PQRD*, either increaſed by the curvilinear area *DFge*, if the place *e* is below the place *D*, or diminiſhed by the ſame area *DFge* if it be higher, is to the right line whoſe power is the rectangle *PQRD* alone.

COR. 3. The time is alſo known by erecting the ordinate *em* reciprocally proportional to the ſquare root of *PQRD* + or — *DFge*, and taking the time in which the body has deſcribed the line *De*, to the time in which another body has fallen with an uniform force from P, and in falling arrived at D, in the proportion of the curvilinear area $DLme$ to the rectangle $2PD \times DL$. For the time in which a body falling with an uniform force hath deſcribed the line PD, is to the time in which the ſame body has deſcribed the line PE, in the ſubduplicate ratio of PD to PE; that is (the very ſmall line DE being juſt naſcent) in the ratio of PD to $PD + \frac{1}{2}DE$, or $2PD$ to $2PD + DE$, and by diviſion to the time in which the body hath deſcribed the ſmall line DE, as $2PD$ to DE, and therefore as the rectangle $2PD \times DL$ to the area $DLME$; and the time in which both the bodies deſcribed the very ſmall line DE is to the time in which the body moving unequably hath deſcribed the line De, as the area $DLME$ to the area $DLme$; and *ex æquo* the firſt mentioned of theſe times is to the laſt as the rectangle $2PD \times DL$ to the area $DLme$.

SECTION VIII.

Of the invention of orbits wherein bodies will revolve, being acted upon by any ſort of centripetal force.

PROPOSITION XL. THEOREM XIII.

If a body, acted upon by any centripetal force, is any how moved, and another body aſcends or deſcends in a right line; and their velocities be equal in any one caſe of equal altitudes, their velocities will be alſo equal at all equal altitudes.

Let a body deſcend from *A* (*Pl.* 17. *Fig.* 3.) through *D* and *E*, to the centre *C*; and let another body move from *V* in the curve line *VIKk*. From the centre *C*, with any diſtances, deſcribe the concentric circles *DI*, *EK*, meeting the right line *AC* in *D* and *E*, and the curve *VIK* in *I* and *K*.

Draw *IC* meeting *KE* in *N*, and on *IK* let fall the perpendicular *NT*; and let the interval *DE* or *IN*, between the circumferences of the circles be very ſmall; and imagine the bodies in *D* and *I* to have equal velocities. Then becauſe the diſtances *CD* and *CI* are equal, the centripetal forces in *D* and *I* will be alſo equal. Let thoſe forces be expreſs'd by the equal lineolæ *DE* and *IN*; and let the force *IN* (by cor 2. of the laws of motion) be reſolved into two others, *NT* and *IT*. Then the force *NT* acting in the direction of the line *NT* perpendicular to the path *ITK* of the body, will not at all affect or change the velocity of the body in that path, but only draw it aſide from a rectilinear courſe, and make it deflect perpetually from the tangent of the orbit, and proceed in the curvilinear path *ITKk*. That whole force therefore will be ſpent in producing this effect; but the other force *IT*, acting in the direction of the courſe of the body, will be all employed in accelerating it; and in the leaſt given time will produce an acceleration proportional to it ſelf. Therefore the accelerations of the bodies in *D* and *I* produced in equal times, are as the lines *DE*, *IT*; (if we take the firſt ratio's of the naſcent lines *DE*, *IN*, *IK*, *IT*, *NT*;) and in unequal times as thoſe lines and the times conjunctly. But the times in which *DE* and *IK* are deſcribed, are, by reaſon of the equal velocities (in *D* and *I*) as the ſpaces deſcribed *DE* and *IK*, and therefore the accelerations in the courſe of the bodies through the lines *DE* and *IK*, are as *DE* and *IT*, and *DE* and *IK* conjunctly; that is, as the ſquare of *DE* to the rectangle *IT* into *IK*. But the rectangle *IT*×*IK* is equal to the ſquare of *IN*, that is equal to the ſquare of *DE*, and therefore the accelerations generated in the paſſage of the bodies from *D* and *I* to *E* and *K* are equal. Therefore the velocities of the bodies in *E* and *K* are alſo equal: and by the ſame reaſoning they will always be found equal in any ſubſequent equal diſtances. *Q. E. D.*

By the ſame reaſoning, bodies of equal velocities and equal diſtances from the centre will be equally retarded in their aſcent to equal diſtances. *Q. E. D.*

Cor. 1. Therefore if a body either oſcillates by hanging to a ſtring, or by any poliſhed and perfectly ſmooth impediment is forced to move in a curve line; and another body aſcends or deſcends in a right line, and their velocities be equal at any one equal altitude; their velocities will be alſo equal at all other equal altitudes. For, by the ſtring of the pendulous body, or by the impediment of a veſſel perfectly ſmooth, the ſame thing will be effected, as by the tranſverſe force *NT*. The body is neither accelerated nor retarded by it, but only is obliged to quit its rectilinear courſe.

Cor. 2. Suppoſe the quantity P to be the greateſt diſtance from the centre to which a body can aſcend, whether it be oſcillating, or revolving in a trajectory, and ſo the ſame projected upwards from any point of a trajectory with the velocity it has in that point. Let the quantity A be the diſtance of the body from the centre in any other point of the orbit; and let the centripetal force be always as the power A^{n-1} of the quantity A, the index of which power $n-1$, is any number n diminiſhed by unity. Then the velocity in every altitude A will be as $\sqrt{P^n - A^n}$, and therefore will be given. For by prop. 39. the velocity of a body aſcending and deſcending in a right line is in that very ratio.

PROPOSITION XLI. PROBLEM XXVIII.

Suppoſing a centripetal force of any kind, and granting the quadratures of curvilinear figures, it is required to find, as well the trajectories in which bodies will move, as the times of their motions in the trajectories found.

Let any centripetal force tend to the centre *C*, (*Pl.* 17. *Fig.* 4.) and let it be required to find the trajectory *VIKk*. Let there be given the circle *VR*, deſcribed from the centre *C* with any interval *CV*; and from the ſame centre deſcribe any other circles *ID*, *KE* cutting the trajectory in *I* and *K*, and the right line *CV* in *D* and *E*. Then draw the right line *CNIX* cutting the circles *KE*, *VR* in *N* and *X*, and the right line *CKY* meeting the circle *VR* in *Y*. Let the points *I* and *K* be indefinitely near; and let the body go on from *V* through *I* and *K* to *k*; and let the point *A* be the place from whence another body is to fall, ſo as in the place *D* to acquire a velocity equal to the velocity of the firſt body in *I*. And things remaining as in prop. 39. the lineola *IK*, deſcribed in the leaſt given time will be as the velocity, and therefore as the right line whoſe power is the area *ABFD*, and the triangle *ICK* proportional to the time will be given, and therefore *KN* will be reciprocally as the altitude *IC*; that is (if there be given any quantity Q and the altitude *IC* be called *A*) as $\frac{Q}{A}$. This quantity $\frac{Q}{A}$ call Z, and ſuppoſe the magnitude of Q to be ſuch that in ſome caſe $\sqrt{ABFD}$ may be to Z as *IK* to *KN*, and then in all caſes $\sqrt{ABFD}$ will be to Z as *IK* to *KN*, and *ABFD* to ZZ as IK^2 to KN^2, and by diviſion $ABFD - ZZ$ to ZZ as IN^2 to KN^2, and therefore $\sqrt{ABFD - ZZ}$ to Z or $\frac{Q}{A}$ as *IN* to *KN*, and therefore $A \times KN$ will be equal to $\frac{Q \times IN}{\sqrt{ABFD - ZZ}}$. Therefore ſince $YX \times XC$ is to $A \times KN$ as CX^2 to AA the rectangle $XY \times XC$ will be equal to $\frac{Q \times IN \times CX^2}{AA\sqrt{ABFD - ZZ}}$. Therefore in the perpendicular *DF* let there be taken continually *Db*, *Dc* equal to $\frac{Q}{2\sqrt{ABFD - ZZ}}$, $\frac{Q \times CX^2}{2AA\sqrt{ABFD - ZZ}}$ reſpectively, and let the curve lines *ab*, *ac*, the loci of the points *b* and *c*, be deſcribed: and from the point *V*, let the perpendicular *Va* be erected to the line *AC*, cutting off the curvilinear area's *VDba*, *VDca*, and let the ordinates *Ez*, *Ex*, be erected alſo. Then becauſe the rectangle $Db \times IN$ or *DbzE* is equal to half the rectangle $A \times KN$ or to the triangle *ICK*; and the rectangle $Dc \times IN$ or *DcxE* is equal to half the rectengle $YX \times XC$ or to the triangle *XCY*; that is, becauſe the naſcent particles *DbzE*, *ICK* of the area's *VDba*, *VIC* are always equal; and the naſcent particles *DcxE*, *XCY* of the area's *VDca*, *VCX* are always equal; therefore the generated area *VDba* will be equal to the generated area *VIC*, and therefore proportional to the time; and the generated area *VDca* is equal to the generated ſector *VCX*. If therefore any time be given during which the body has been moving from *V*, there will be alſo given the area proportional to it *VDba*; and thence will be given the altitude of the body *CD* or *CI*; and the area *VDca*, and the ſector *VCX* equal thereto, together with its angle *VCI*. But the angle *VCI*, and the altitude *CI* being given, there is alſo given the place *I*, in which the body will be found at the end of that time. *Q. E. I.*

COR. 1. Hence the greateſt and leaſt altitudes of the bodies, that is the apſides of the trajectories, may be found very readily. For the apſides are thoſe points in which a right line *IC* drawn thro' the centre falls perpendicularly upon the trajectory *VIK*; which comes to paſs when the right lines *IK* and *NK* become equal; that is, when the area *ABFD* is equal to ZZ.

COR. 2. So alſo, the angle *KIN* in which the trajectory at any place cuts the line *IC*, may be readily found by the given altitude *IC* of the body: to wit, by making the ſine of that angle to radius as *KN* to *IK*; that is as Z to the ſquare root of the area *ABFD*.

COR. 3. If to the centre *C* (*Pl.* 17. *Fig.* 5.) and the principal vertex *V* there be deſcribed a conic ſection *VRS*; and from any point thereof as *R*, there be drawn the tangent *RT* meeting the axe *CV* indefinitely produced, in the point *T*;

T; and then, joining *CR*, there be drawn the right line *CP*, equal to the abſciſſa *CT*, making an angle *VCP* proportional to the ſector *VCR*; and if a centripetal force, reciprocally proportional to the cubes of the diſtances of the places from the centre, tends to the centre *C*; and from the place *V* there ſets out a body with a juſt velocity in the direction of a line perpendicular to the right line *CV*: that body will proceed in a trajectory *VPQ*, which the point *P* will always touch; and therefore if the conic ſection *VRS* be an hyperbola, the body will deſcend to the centre; but if it be an ellipſis it will aſcend perpetually, and go farther and farther off in infinitum. And on the contrary, if a body endued with any velocity goes off from the place *V*, and according as it begins either to deſcend obliquely to the centre or aſcends obliquely from it, the figure *VRS* be either an hyperbola or an ellipſis, the trajectory may be found by increaſing or diminiſhing the angle *VCP* in a given ratio. And the centripetal force becoming centrifugal, the body will aſcend obliquely in the trajectory *VPQ*, which is found by taking the angle *VCP* proportional to the elliptic ſector *VRC*, and the length *CP* equal to the length *CT*, as before. All theſe things follow from the foregoing propoſition, by the quadrature of a certain curve, the invention of which, as being eaſy enough, for brevity's ſake I omit.

PRO-

PROPOSITION XLII. PROBLEM XXIX.

The law of centripetal force being given, it is required to find the motion of a body ſetting out from a given place, with a given velocity, in the direction of a given right line.

Suppoſe the ſame things as in the three preceding propoſitions; and let the body go off from the place *I*, (*Pl.* 17. *Fig.* 6.) in the direction of the little line *IK*, with the ſame velocity as another body, by falling with an uniform centripetal force from the place *P*, may acquire in *D*; and let this uniform force be to the force with which the body is at firſt urged in *I*, as *DR* to *DF*. Let the body go on towards *k*; and about the centre *C* with the interval *Ck*, deſcribe the circle *ke*, meeting the right line *PD* in *e*, and let there be erected the lines *eg*, *ev*, *ew*, ordinately applied to the curves *BFg*, *abv*, *acw*. From the given rectangle *PDRQ* and the given law of centripetal force, by which the firſt body is acted on, the curve line *BFg* is alſo given, by the conſtruction of prob. 27. and its cor. 1. Then from the given angle *CIK* is given the proportion of the naſcent lines *IK*, *KN*; and thence by the conſtruction of prob. 28. there is given the quantity Q, with the curve lines *abv*, *acw*; and therefore, at the end of any time *Dbve*, there is given both the altitude of the body *Ce* or *Ck*, and the area *Dcwe*, with the ſector equal to it *XCy*, the angle *ICk*, and the place *k*, in which the body will then be found. *Q. E. I.*

We ſuppoſe in theſe propoſitions the centripetal force to vary in its receſs from the centre according to ſome law, which any one may imagine at pleaſure; but at equal diſtances from the centre to be every where the ſame.

I have hitherto conſidered the motions of bodies in immoveable orbits. It remains now to add ſomething concerning their motions in orbits which revolve round the centres of force.

Plate XVII. *Vol.* I. *P.* 176.

Fig. 1. *p.* 162.

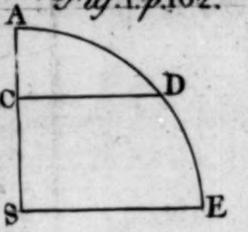


Fig. 2. *p.* 163.

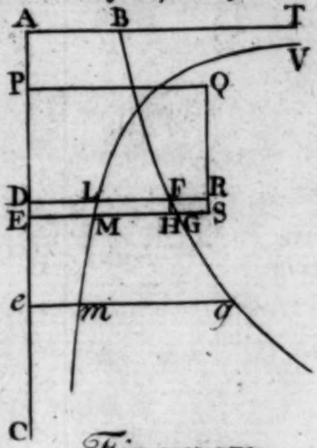


Fig. 3. *p.* 168.

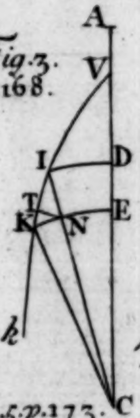


Fig. 4. *p.* 171.

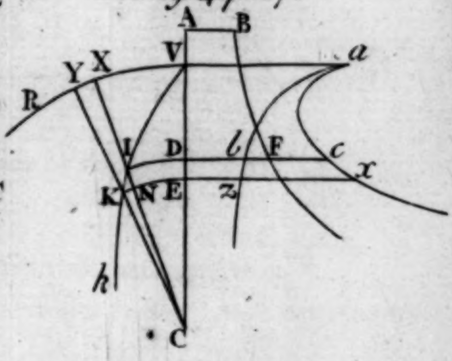


Fig. 5. *p.* 173.

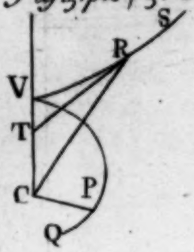


Fig. 6. *p.* 175.

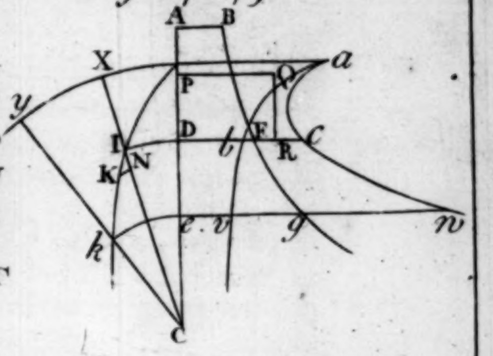
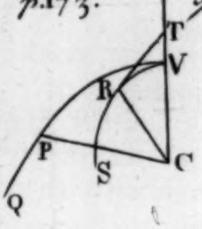


Fig. 5. *p.* 173.



SECTION IX.

Of the motion of bodies in moveable orbits; and of the motion of the apſides.

PROPOSITION XLIII. PROBLEM XXX.

It is required to make a body move, in a trajectory that revolves about the centre of force, in the ſame manner as another body in the ſame trajectory at reſt.

In the orbit *VPK* (*Pl.* 18. *Fig.* 1.) given by poſition, let the body *P* revolve, proceeding from *V* towards *K*. From the centre *C* let there be continually drawn *Cp*, equal to *CP*, making the angle *VCp* proportional to the angle *VCP*; and the area which the line *Cp* deſcribes, will be to the area *VCP* which the line *CP* deſcribes at the ſame time, as the velocity of the deſcribing line *Cp*,

to

to the velocity of the deſcribing line *CP*; that is, as the angle *VCp* to the angle *VCP*, therefore in a given ratio, and therefore proportional to the time. Since then the area deſcribed by the line *Cp* in an immoveable plane is proportional to the time, it is manifeſt that a body, being acted upon by a juſt quantity of centripetal force, may revolve with the point *p* in the curve line which the ſame point *p*, by the method juſt now explained, may be made to deſcribe in an immoveable plane. Make the angle *VCu* equal to the angle *PCp*, and the line *Cu* equal to *CV*, and the figure *uCp* equal to the figure *VCP*, and the body being always in the point *p*, will move in the perimeter of the revolving figure *uCp*, and will deſcribe its (revolving) arc *up* in the ſame time that the other body *P* deſcribes the ſimilar and equal arc *VP* in the quieſcent figure *VPK*. Find then by cor. 5. prop. 6. the centripetal force by which a body may be made to revolve in the curve line which the point *p* deſcribes in an immoveable plane, and the problem will be ſolved. *Q. E. F.*

PROPOSITION XLIV. THEOREM XIV.

The difference of the forces, by which two bodies may be made to move equally, one in a quieſcent, the other in the ſame orbit revolving, is in a triplicate ratio of their common altitudes inverſely.

Let the parts of the quieſcent orbit *VP*, *PK*, (*Pl.* 18. *Fig.* 2.) be ſimilar and equal to the parts of the revolving orbit *up*, *pk*; and let the diſtance of the points *P* and *K* be ſuppoſed of the utmoſt ſmallneſs. Let fall a perpendicular *kr* from the point *k* to the right line *pC*, and produce it to *m*, ſo that *mr* may be to *kr* as the angle *VCp* to the angle *VCP*. Becauſe the altitudes of the bodies, *PC* and *pC*, *KC* and *kC*, are always equal, it is manifeſt that the increments or decrements of the lines *PC* and *pC* are always equal; and therefore if each of the ſeveral motions of the bodies in the places *P* and *p* be reſolved into two, (by cor. 2. of the laws of motion) one of which is directed towards the center, or according to the lines *PC*, *pC*, and the other, tranſverſe to the former, hath a direction perpendicular to the lines *PC* and *pC*; the motions towards the centre will be equal, and the tranſverſe motion of the body *p* will be to the tranſverſe motion of the body *P*, as the angular motion of the line *pC* to the angular motion of the line *PC*; that is, as the angle *VCp* to the angle *VCP*. Therefore at the ſame time that the body *P*, by both its motions, comes to the point *K*, the body *p*, having an equal motion towards the centre, will be equally moved from *p* towards *C*, and therefore that time being expired, it will be found ſomewhere in the line *mkr*, which, paſſing through the point *k*, is perpendicular to the line *pC*; and by its tranſverſe motion, will acquire a diſtance from the line *pC*, that will be to the diſtance which the other body *P* acquires from the line *PC*, as the tranſverſe motion of the body *p*, to the tranſverſe motion of the other body *P*. Therefore ſince *kr* is equal to the diſtance which the body *P* acquires from the line *PC*, and *mr* is to *kr* as the angle VCp to the angle VCP, that is as the tranſverſe motion of the body p, to the tranſverſe motion of the body P; it is manifeſt that the body p, at the expiration of that time, will be ſound in the place m. Theſe things will be ſo, if the bodies p and P are equally moved in the directions of the lines pC and PC, and are therefore urged with equal forces in thoſe directions. But if we take an angle pCn that is to the angle pCk as the angle VCp to the angle VCP, and nC be equal to kC, in that caſe the body p at the expiration of the time will really be in n; and is therefore urged with a greater force than the body P, if the angle nCp is greater than the angle kCp, that is, if the orbit upk move either *in conſequentia*, or *in antecedentia* with a celerity greater than the double of that with which the line CP moves *in conſequentia*; and with a leſs force if the orbit moves ſlower in *antecedentia*. And the difference of the forces will be as the interval mn of the places through which the body would be carried by the action of that difference in that given ſpace of time. About the centre C with the interval Cn or Ck ſuppoſe a circle deſcribed cutting the lines mr, mn produced in s and t, and the rectangle $mn \times mt$ will be equal to the rectangle $mk \times ms$, and therefore mn will be equal to $\frac{mk \times ms}{mt}$. But ſince the triangles pCk, pCn, in a given time, are of a given magnitude, kr and mr, and their difference mk, and their ſum ms, are reciprocally as the altitude pC, and therefore the rectangle $mk \times ms$ is reciprocally as the ſquare of the altitude pC. But moreover mt is directly as $\frac{1}{2}mt$, that is, as the altitude *pC*. Theſe are the firſt ratio's of the naſcent lines; and hence $\frac{mk \times ms}{mt}$, that is, the naſcent lineola *mn*, and the difference of the forces proportional thereto, are reciprocally as the cube of the altitude *pC*. *Q. E. D.*

Cor. 1. Hence the difference of the forces in the places *P* and *p*, or *K* and *k*, is to the force with which a body may revolve with a circular motion from *R* to *K*, in the ſame time that the body *P* in an immoveable orb deſcribes the arc *PK*, as the naſcent line *mn* to the verſed ſine of the naſcent arc *RK*, that is as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$, or as $mk \times ms$ to the ſquare of *rk*; that is, if we take given quantities F and G in the ſame ratio to one another as the angle *VCP* bears to the angle *VCp*, as GG—FF to FF. And therefore if from the centre *C* with any diſtance *CP* or *Cp*, there be deſcribed a circular ſector equal to the whole area *VPC*, which the body revolving in an immoveable orbit, has by a radius drawn to the centre deſcribed in any certain time; the difference of the forces, with which the body *P* revolves in an immoveable orbit and the body *p* in a moveable orbit, will be to the centripetal force, with which another body by a radius drawn to the centre can uniformly deſcribe that ſector in the ſame time as the area *VPC* is deſcribed, as GG—FF to FF. For that ſector and the area *pCk* are to one another as the times in which they are deſcribed.

Cor. 2. If the orbit *VPK* be an ellipſis having its focus *C*, and its higheſt apſis *V*, and we ſuppoſe the ellipſis *upk* ſimilar and equal to it, ſo that *pC* may be always equal to *PC*, and the angle *VCp* be to the angle *VCP* in the given ratio of G to F; and for the altitude *PC* or *pC* we put A, and 2 R for the latus rectum of the ellipſis; the force with which a body may be made to revolve in a moveable ellipſis will be as $\frac{\text{FF}}{\text{AA}} + \frac{\text{RGG} - \text{RFF}}{\text{A}^3}$ and vice verſa. Let the force with which a body may revolve in an immoveable ellipſis, be expreſſed by the quantity $\frac{\text{FF}}{\text{AA}}$, and the force in V will be $\frac{\text{FF}}{CV^2}$. But the force with which a body may revolve in a circle at the diſtance *CV*, with the ſame velocity as a body revolving in an ellipſis has in *V*, is to the force with which a body revolving in an ellipſis is acted upon in the apſis *V*, as half the latus rectum of the ellipſis, to the ſemi-diameter *CV* of the circle, and therefore is as $\frac{\text{RFF}}{CV^3}$; and the force which is to this as GG—FF to FF, is as $\frac{\text{RGG} - \text{RFF}}{CV^3}$: and this force (by cor. 1. of this prop.) is the difference of the forces in *V*, with which the body *P* revolves in the immoveable ellipſis *VPK*, and the body *p* in the moveable ellipſis *upk*. Therefore ſince by this prop. that difference at any other altitude A is to it ſelf at the altitude *CV* as $\frac{1}{\text{A}^3}$ to $\frac{1}{CV^3}$, the ſame difference in every altitude A will be as $\frac{RGG-RFF}{A^3}$. Therefore to the force $\frac{FF}{AA}$, by which the body may revolve in an immoveable ellipſis *VPK*, add the exceſs $\frac{RGG-RFF}{A^3}$ and the ſum will be the whole force $\frac{FF}{AA} + \frac{RGG-RFF}{A^3}$ by which a body may revolve in the ſame time in the moveable ellipſis ***upk***.

Cor. 3. In the ſame manner it will be found that if the immoveable orbit *VPK* be an ellipſis having its centre in the centre of the forces *C*; and there be ſuppoſed a moveable ellipſis ***upk***, ſimilar, equal, and concentrical to it; and 2*R* be the principal latus rectum of that ellipſis, and 2T the latus tranſverſum or greater axis; and the angle *VCp* be continually to the angle *VCP* as G to F; the forces with which bodies may revolve in the immoveable and moveable ellipſis in equal times, will be as $\frac{FFA}{T^3}$ and $\frac{FFA}{T^3} + \frac{RGG-RFF}{A^3}$ reſpectively.

Cor. 4. And univerſally, if the greateſt altitude *CV* of the body be called T, and the radius of the curvature which the orbit *VPK* has in *V*, that is, the radius of a circle equally curve, be called R, and the centripetal force with which a body may revolve in any immoveable trajectory *VPK* at the place *V*, be called $\frac{VFF}{TT}$, and in other places *P* be indefinitely ſtiled X; and the altitude *CP* be called

A, and G be taken to F in the given ratio of the angle *VCp* to the angle *VCP*: the centripetal force with which the ſame body will perform the ſame motions in the ſame time in the ſame trajectory *upk* revolving with a circular motion, will be as the ſum of the forces $X + \frac{VRGG - VRFF}{A^3}$.

Cor. 5. Therefore the motion of a body in an immoveable orbit being given, its angular motion round the centre of the forces may be increaſed or diminiſhed in a given ratio, and thence new immoveable orbits may be ſound in which bodies may revolve with new centripetal forces.

Cor. 6. Therefore if there be erected (*Pl.* 18. *Fig.* 3.) the line *VP* of an indeterminate length, perpendicular to the line *CV* given by poſition, and *CP* be drawn, and *Cp* equal to it, making the angle *VCp* having a given ratio to the angle *VCP*; the force with which a body may revolve in the curve line *Vpk*, which the point *p* is continually deſcribing, will be reciprocally as the cube of the altitude *Cp*. For the body *P*, by its vis inertiæ alone, no other force impelling it, will proceed uniformly in the right line *VP*. Add then a force tending to the centre *C* reciprocally as the cube of the altitude *CP* or *Cp*, and (by what was juſt demonſtrated) the body will deflect from the rectilinear motion into the curve line *Vpk*. But this curve *Vpk* is the ſame with the curve *VPQ* found in cor. 3. prop. 41. in which, I ſaid, bodies attracted with ſuch forces would aſcend obliquely.

Proposition XLV. Problem XXXI.

To find the motion of the apſides in orbits approaching very near to circles.

This problem is ſolved arithmetically by reducing the orbit, which a body revolving in a moveable ellipſis (as in cor. 2 and 3 of the above prop.) deſcribes in an immoveable plane, to the figure of the orbit whoſe apſides are required; and then ſeeking the apſides of the orbit which that body deſcribes in an immoveable plane. But orbits acquire the ſame figure, if the centripetal forces with which they are deſcribed, compared between themſelves, are made proportional at equal altitudes. Let the point *V* be the higheſt apſis, and write T for the greateſt altitude *CV*, A for any other altitude *CP* or *Cp*, and X for the difference of the altitudes *CV*—*CP*; and the force with which a body moves in an ellipſis revolving about its focus *C* (as in cor. 2.) and which in cor. 2. was as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, that is, as $\frac{FFA + RGG - RFF}{A^3}$, by ſubſtituting $T - X$ for A will become as $\frac{RGG - RFF + TFF - FFX}{A^3}$. In like manner any other centripetal force is to be reduced to a fraction whoſe denominator is A^3 and the numerators are to be made analogous by collating together the homologous terms. This will be made plainer by examples.

Exam. 1. Let us ſuppoſe the centripetal force to be uniform, and therefore as $\frac{A^3}{A^3}$, or, writing T—X for A in the numerator, as $\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$ $\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$. Then collating together the correſpondent terms of the numerators, that is, thoſe that conſiſt of given quantities, with thoſe of given quantities, and thoſe of quantities not given, with thoſe of quantities not given, it will become $RGG - RFF + TFF$ to T^3 as $-FFX$ to $-3TTX + 3TXX - X^3$ or as $-FF$ to $-3TT + 3TX - XX$. Now ſince the orbit is ſuppoſed extreamly near to a circle, let it coincide with a circle, and becauſe in that caſe R and T become equal, and X is infinitely diminiſhed, the laſt ratio's will be, as RGG to T^3 ſo $-FF$ to $-3TT$, or as GG to TT ſo FF to 3TT, and again as GG to FF ſo TT to 3TT, that is, as 1 to 3; and therefore G is to F, that is, the angle *VCp* to the angle *VCP* as 1 to $\sqrt{3}$. Therefore ſince the body, in an immoveable ellipſis, in deſcending from the upper to the lower apſis, deſcribes an angle, if I may ſo ſpeak, of 180 deg. the other body in a moveable ellipſis, and therefore in the immoveable orbit we are treating of, will, in its deſcent from the upper to the lower apſis, deſcribe an angle *VCp* of $\frac{180}{\sqrt{3}}$ deg. And this comes to paſs by reaſon of the likeneſs of this orbit which a body acted upon by an uniform centripetal force deſcribes, and of that orbit which a body performing its circuits in a revolving ellipſis will deſcribe in a quieſcent plane. By this collation of the terms, theſe orbits are made ſimilar; not univerſally indeed, but then only when they approach very near to a circular figure. A body therefore revolving with an uniform centripetal force in an orbit nearly circular, will always deſcribe an angle of $\frac{180}{\sqrt{3}}$ deg. or 103 deg. 55 m. 23 ſec. at the centre; moving from the upper apſis to the lower apſis when it has once deſcribed that angle, and thence returning to the upper apſis when it has deſcribed that angle again; ann ſo on in infinitum.

Exam. 2. Suppoſe the centripetal force to be as any power of the altitude A, as for example A^{n-3} or $\frac{A^n}{A^3}$; where $n-3$ and n ſignify any indices of powers whatever, whether integers or fractions, rational or ſurd, affirmative or negative. That numerator A^n or $\overline{T-X}|^n$ being reduced to an indeterminate ſeries by my method of converging ſeries, will become $T^n - nXT^{n-1} + \frac{nn-n}{2}XXT^{n-2}$ *&c.* And conferring theſe terms with the terms of the other numerator $RGG - RFF + TFF - FFX$, it becomes as $RGG - RFF + TFF$ to T^n ſo $-FF$ to $-nT^{n-1} + \frac{nn-n}{2}XT^{n-2}$ *&c.* And taking the laſt ratio's where the orbits approach to circles, it becomes as RGG to T^n ſo $-FF$ to $-nT^{n-1}$, or as GG to T^{n-1} ſo FF to nT^{n-1} and again GG to FF ſo T^{n-1} to nT^{n-1}, that is, as 1 to n; and therefore G is to F, that is the angle *VCp* to the angle *VCP* as 1 to $\sqrt{n}$. Therefore ſince the angle *VCP*, deſcribed in the deſcent of the body from the upper apſis to the lower apſis in an ellipſis, is of 180 deg. the angle *VCp*, deſcribed in the deſcent of the body from the upper apſis to the lower apſis in an orbit nearly circular which a body deſcribes with a centripetal force proportional to the power A^{n-3}, will be equal to an angle of $\frac{180}{\sqrt{n}}$ deg. and this angle being repeated the body will return from the lower to the upper apſis, and ſo on in infinitum. As if the centripetal force be as the diſtance of the body from the centre, that is, as A, or $\frac{A^4}{A^3}$, n will be equal to 4, and $\sqrt{n}$ equal to 2; and therefore the angle between the upper and the lower apſis will be equal to $\frac{180}{2}$ deg. or 90 deg. Therefore the body having performed a fourth part of one revolution will arrive at the lower apſis, and having performed another fourth part, will arrive at the upper apſis, and ſo on by turns in infinitum. This appears alſo from prop. 10. For a body acted on by this centripetal force will revolve in an immoveable ellipſis, whoſe centre is the centre of force. If the centripetal force is reciprocally as the diſtance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n will be equal to 2, and therefore the angle between the upper and lower apſis will be $\frac{180}{\sqrt{2}}$ deg. or 127 deg. 16 min. 45 ſec. and therefore a body revolving with ſuch a force, will, by a perpetual repetition of this angle, move alternately from the upper to the lower, and from the lower to the upper apſis for ever. So alſo if the centripetal force be reciprocally as the biquadrate root of the eleventh power of the altitude, that is reciprocally as $A^{\frac{11}{4}}$ and therefore directly as $\frac{1}{A^{\frac{11}{4}}}$ or as $\frac{A^{\frac{1}{4}}}{A^3}$, n will be equal to $\frac{1}{4}$, and $\frac{180}{\sqrt{n}}$ deg. will be equal to 360 deg. and therefore the body parting from the upper apſis, and from thence perpetually deſcending will arrive at the lower apſis when it has compleated one entire revolution; and thence aſcending perpetually, when it has compleated another entire revolution it will arrive again at the upper apſis; and ſo alternately for ever.

Exam. 3. Taking m and n for any indices of the powers of the altitude, and b and c for any given numbers, ſuppoſe the centripetal force to be as $\frac{bA^m + cA^n}{A^3}$ that is, as $\frac{b \text{ into } \overline{T-X}|^m + c \text{ into } \overline{T-X}|^n}{A^3}$ or (by the method of converging ſeries above-mentioned) as $\frac{bT^m + cT^n - mbXT^{m-1} - ncXT^{n-1}}{A^3}$ $+ \frac{mm-m}{2}bXXT^{m-2} + \frac{nn-n}{2}cXXT^{n-2}$ *&c.* and comparing the terms of the numerators, there will ariſe $RGG - RFF + TFF$ to $bT^m + cT^n$ as $-FF$ to $-mbT^{m-1} - ncT^{n-1} + \frac{mm-m}{2}bXT^{m-2} + \frac{nn-n}{2}cXT^{n-2}$ &c. And taking the laſt ratio's that ariſe when the orbits come to a circular form, there will come forth GG to $bT^{m-1} + cT^{n-1}$ as FF to $mbT^{m-1} + ncT^{n-1}$, and again GG to FF as $bT^{m-1} + cT^{n-1}$ to $mbT^{n-1} + ncT^{n-1}$. This proportion, by expreſſing the greateſt altitude *CV* or T arithmetically by unity, becomes, GG to FF as $b+c$ to $mb+nc$, and therefore as 1 to $\frac{mb+nc}{b+c}$. Whence G becomes to F, that is the angle *VCp* to the angle *VCP* as 1 to $\sqrt{\frac{mb+nc}{b+c}}$. And therefore ſince the angle *VCP* between the upper and the lower apſis, in an immoveable ellipſis, is of 180 deg. the angle *VCp* between the ſame apſides in an orbit which a body deſcribes with a centripetal force, that is as $\frac{bA^m + cA^n}{A^3}$ will be equal to an angle of $180\sqrt{\frac{b+c}{mb+nc}}$ deg. And by the ſame reaſoning if the centripetal force be as $\frac{bA^m - cA^n}{A^3}$ the angle between the apſides will be found equal to $180\sqrt{\frac{b-c}{mb-nc}}$ deg. After the ſame manner the problem is ſolved in more difficult caſes. The quantity to which the centripetal force is proportional, muſt always be reſolved into a converging ſeries whoſe denominator is A^3. Then the given part of the numerator ariſing from that operation is to be ſuppoſed in the ſame ratio to that part of it which is not given, as the given part of this numerator $RGG - RFF + TFF - FFX$ is to that part of the ſame numerator which is not given. And taking away the ſuperfluous quantities and writing unity for T, the proportion of G to F is obtained.

Cor. 1. Hence if the centripetal force be as any power of the altitude, that power may be found from the motion of the apſides; and ſo contrarywiſe. That is, if the whole angular motion, with which the body returns to the ſame apſis, be to the angular motion of one revolution, or 360 deg. as any number as m to another as n, and the altitude called A; the force will be as the power $A\frac{nn}{mm}-3$ of the altitude A; the index of which power is $\frac{nn}{mm}-3$. This appears by the ſecond examples. Hence 'tis plain that the force in its receſs from the centre cannot decreaſe in a greater than a triplicate ratio of the altitude. A body revolving with ſuch a force and parting from the apſis, if it once begins to deſcend can never arrive at the lower apſis or leaſt altitude, but will deſcend to the centre, deſcribing the curve line treated of in cor. 3. prop. 41. But if it ſhould, at its parting from the lower apſis begin to aſcend never ſo little, it will aſcend in infinitum and never come to the upper apſis; but will deſcribe the curve line ſpoken of in the ſame cor. and cor. 6. prop. 44. So that where the force in its receſs from the centre decreaſes in a greater than a triplicate ratio of the altitude, the body at its parting from the apſis, will either deſcend to the centre or aſcend in infinitum, according as it deſcends or aſcends at the beginning of its motion. But if the force in its receſs from the centre either decreaſes in a leſs than a triplicate ratio of the altitude, or increaſes in any ratio of the altitude whatſoever; the body will never deſcend to the centre, but will at ſome time arrive at the lower apſis; and on the contrary, if the body alternately aſcending and deſcending from one apſis to another never comes to the centre, then either the force increaſes in the receſs from the centre, or it decreaſes in a leſs than a triplicate ratio of the altitude; and the ſooner the body returns from one apſis to another, the farther is the ratio of the forces from the triplicate ratio. As if the body ſhould return to and from the upper apſis by an alternate deſcent and aſcent in 8 revolutions, or in 4, or 2, or $1\frac{1}{2}$; that is if *m* ſhould be to *n* as 8 or 4 or 2 or $1\frac{1}{2}$ to 1, and therefore $\frac{nn}{mm}-3$ be $\frac{1}{64}-3$, or $\frac{1}{16}-3$, or $\frac{1}{4}-3$, or $\frac{4}{9}-3$; then the force will be as $A^{\frac{1}{64}-3}$, or $A^{\frac{1}{16}-3}$, or $A^{\frac{1}{4}-3}$, or $A^{\frac{4}{9}-3}$; that is, it will be reciprocally as $A^{3-\frac{1}{64}}$, or $A^{3-\frac{1}{16}}$, or $A^{3-\frac{1}{4}}$, or $A^{3-\frac{4}{9}}$. If the body after each revolution returns to the ſame apſis, and the apſis remains unmoved, then *m* will be to *n* as 1 to 1, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to A^{-2} or $\frac{1}{AA}$; and therefore the decreaſe of the forces will be in a duplicate ratio of the altitude; as was demonſtrated above. If the body in three fourth parts, or two thirds, or one third, or one fourth part of an entire revolution, return to the ſame apſis; m will be to n as $\frac{3}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and therefore $A^{\frac{nn}{mm}-3}$ is equal to $A^{\frac{16}{9}-3}$, or $A^{\frac{9}{4}-3}$, or A^{9-3}, or A^{16-3}; and therefore the force is either reciprocally as $A^{\frac{11}{9}}$ or $A^{\frac{3}{4}}$, or directly as A^{6} or A^{13}. Laſtly, if the body in its progreſs from the upper apſis to the ſame upper apſis again, goes over one entire revolution and three deg. more, and therefore that apſis in each revolution of the body moves three deg. in conſequentia; then m will be to n as 363 deg. to 360 deg. or as 121 to 120, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to $A^{-\frac{29523}{14641}}$, and therefore the centripetal force will be reciprocally as $A^{\frac{29523}{14641}}$, or reciprocally as $A^{2\frac{4}{243}}$ very nearly. Therefore the centripetal force decreaſes in a ratio ſomething greater than the duplicate; but approaching $59\frac{3}{4}$ times nearer to the duplicate than the triplicate.

Cor. 2. Hence alſo if a body, urged by a centripetal force which is reciprocally as the ſquare of the altitude, revolves in an ellipſis whoſe focus is in the centre of the forces; and a new and foreign force ſhould be added to or ſubducted from this centripetal force, the motion of the apſides ariſing from that foreign force may (by the third examples) be known; and ſo on the contrary. As if the force with which the body revolves in the ellipſis be as $\frac{1}{AA}$; and the foreign force ſubducted as cA, and therefore the remaining force as $\frac{A - cA^4}{A^3}$; then (by the third exam.) b will be equal to 1, m equal to 1, and n equal to 4; and therefore the angle of revolution between the apſides is equal to $180\sqrt{\frac{1-c}{1-4c}}$ deg. Suppoſe that foreign force to be 357.45 parts leſs than the other force with which the body revolves in the ellipſis; that is c to be $\frac{100}{35745}$, A or T being equal to 1; and then $180\sqrt{\frac{1-c}{1-4c}}$ will be $180\sqrt{\frac{35645}{35345}}$ or 180.7623, that is, 180 deg. 45 min. 44 ſec. Therefore the body parting from the upper apſis, will arrive at the lower apſis with an angular motion of 180 deg. 45 min. 44 ſec. and this angular motion being repeated will return to the upper apſis; and therefore the upper apſis in each revolution will go forward 1 deg. 31 m. 28 ſec. The apſis of the Moon is about twice as ſwift.

So much for the motion of bodies in orbits whoſe planes paſs through the centre of force. It now remains to determine thoſe motions in eccentrical planes. For thoſe authors who treat of the motion of heavy bodies uſe to conſider the aſcent and deſcent of ſuch bodies, not only in a perpendicular direction, but at all degrees of obliquity upon any given planes; and for the ſame reaſon we are to conſider in this place the motions of bodies tending to centres by means of any forces whatſoever, when thoſe bodies move in eccentrical planes. Theſe planes are ſuppoſed to be perfectly ſmooth and poliſhed ſo as not to retard the motion of the bodies in the leaſt. Moreover in theſe demonſtrations inſtead of the planes upon which thoſe bodies roll or ſlide, and which are therefore tangent planes to the bodies, I ſhall uſe planes parallel to them, in which the centres of the bodies move, and by that motion deſcribe orbits. And by the ſame method I afterwards determine the motions of bodies performed in curve ſuperficies.

SECTION X.

Of the motion of bodies in given ſuperficies, and of the reciprocal motion of funependulous bodies.

PROPOSITION XLVI. PROBLEM XXXII.

Any kind of centripetal force being ſuppoſed, and the centre of force, and any plane whatſoever in which the body revolves, being given, and the quadratures of curvilinear figures being allowed; it is required to determine the motion of a body going off from a given place, with a given velocity, in the direction of a given right line in that plane.

Let *S* (*Pl.* 18. *Fig.* 4.) be the centre of force, *SC* the leaſt diſtance of that centre from the given plane, *P* a body iſſuing from the place *P* in the direction of the right line *PZ*, *Q* the ſame body revolving in its trajectory, and *PQR* the trajectory it ſelf which is required to be found, deſcribed in that given plane. Join *CQ*, *QS*, and if in *QS* we take *SV* proportional to the centripetal force with which the body is attracted towards the centre *S*, and draw *VT* parallel to *CQ*, and meeting *SC* in *T*: then will the force *SV* be reſolved into two, (by cor. 2. of the laws of motion) the force *ST*, and the force *TV*; of which *ST* attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane. But the action of the other force *TV*, coinciding with the poſition of the plane it ſelf, attracts the body directly towards the given point *C* in that plane; and therefore cauſes the body to move in this plane in the ſame manner as if the force *ST* were taken away, and the body were to revolve in free ſpace about the centre *C* by means of the force *TV* alone. But there being given the centripetal force *TV* with which the body *Q* revolves in free ſpace about the given centre *C*, there is given (by prop. 42.) the trajectory *PQR* which the body deſcribes; the place *Q*, in which the body will be found at any given time; and laſtly, the velocity of the body in that place *Q*. And ſo *è contra*. *Q.E.I.*

PROPOSITION XLVII. THEOREM XV.

Suppoſing the centripetal force to be proportional to the diſtance of the body from the centre; all bodies revolving in any planes whatſoever will deſcribe ellipſes, and compleat their revolutions in equal times; and thoſe which move in right lines, running backwards and forwards alternately, will compleat their ſeveral periods of going and returning in the ſame times.

For letting all things ſtand as in the foregoing propoſition, the force *SV*, with which the body *Q* revolving in any plane *PQR* is attracted towards the centre *S*, is as the diſtance *SQ*; and therefore becauſe *SV* and *SQ*, *TV* and *CQ* are proportional, the force *TV* with which the body is attracted towards the given point *C* in the plane of the orbit is as the diſtance *CQ*. Therefore the forces with which bodies found in the plane *PQR* are attracted towards the point *C*, are in proportion to the diſtances equal to the forces with which the ſame bodies are attracted every way towards the centre *S*; and therefore the bodies will move in the ſame times, and in the ſame figures in any plane *PQR* about the point *C*, as they would do in free ſpaces about the centre *S*; and therefore (by cor. 2. prop. 10. and cor. 2. prop.

Plate XVIII. Vol. I. P. 198.

Fig. 1. p. 177.

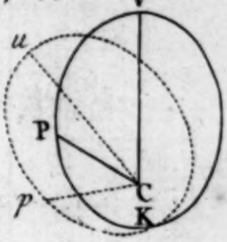


Fig. 2 p. 178.

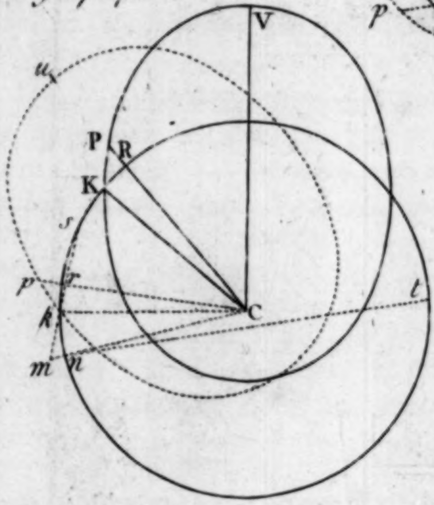


Fig. 3 p. 184.

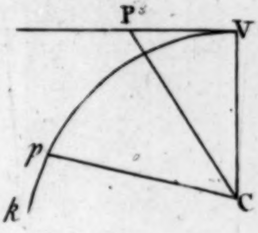
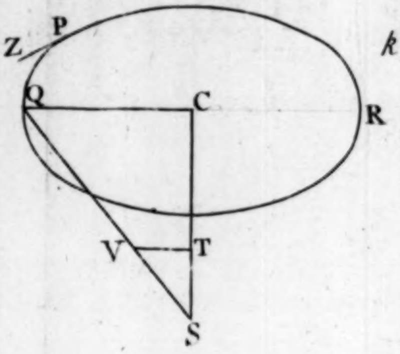


Fig 4 p. 196.



prop. 38) they will in equal times either deſcribe ellipſes in that plane about the centre *C*, or move to and fro in right lines paſſing through the centre *C* in that plane; compleating the ſame periods of time in all caſes. *Q. E. D.*

SCHOLIUM.

The aſcent and deſcent of bodies in curve ſuperficies has a near relation to theſe motions we have been ſpeaking of. Imagine curve lines to be deſcribed on any plane, and to revolve about any given axes paſſing through the centre of force, and by that revolution to deſcribe curve ſuperficies; and that the bodies move in ſuch ſort that their centres may be always found in thoſe ſuperficies. If thoſe bodies reciprocate to and fro with an oblique aſcent and deſcent; their motions will be performed in planes paſſing through the axis, and therefore in the curve lines by whoſe revolution thoſe curve ſuperficies were generated. In thoſe caſes therefore it will be ſufficient to conſider the motion in thoſe curve lines.

PROPOSITION XLVIII. THEOREM XVI.

If a wheel ſtands upon the out-ſide of a globe at right angles thereto, and revolving about its own axis goes forward in a great circle; the length of the curvilinear path which any point, given in the perimeter of the wheel, *hath deſcribed ſince the time that it touched the globe, (which curvilinear path we may call the cycloid or epicycloid) will be to double the verſed ſine of half the arc which ſince that time has touched the globe in paſſing over it, as the ſum of the diameters of the globe and the wheel, to the ſemidiameter of the globe.*

PROPOSITION XLIX. THEOREM XVII.

If a wheel ſtand upon the inſide of a concave globe at right angles thereto, and revolving about its own axis go forward in one of the great circles of the globe, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath deſcribed ſince it touched the globe, will be to the double of the verſed ſine of half the arc which in all that time has touched the globe in paſſing over it, as the difference of the diameters of the globe and the wheel, to the ſemidiameter of the globe.

Let *ABL* (*Pl.* 19. *Fig.* 1. 2.) be the globe, *C* its centre, *BPV* the wheel inſiſting thereon, *E* the centre of the wheel, *B* the point of contact, and *P* the given point in the perimeter of the wheel. Imagine this wheel to proceed in the great circle ABL from A through B towards L, and in its progreſs to revolve in ſuch a manner that the arcs AB, PB may be always equal the one to the other, and the given point P in the perimeter of the wheel may deſcribe in the mean time the curvilinear path AP. Let AP be the whole curvilinear path deſcribed ſince the wheel touched the globe in A, and the length of this path AP will be to twice the verſed ſine of the arc $\frac{1}{2}PB$, as $2CE$ to CB. For let the right line CE (produced if need be) meet the wheel in V, and join CP, BP, EP, VP; produce CP, and let fall thereon the perpendicular VF. Let PH, VH, meeting in H, touch the circle in P and V, and let PH cut VF in G, and to VP let fall the perpendiculars GI, HK. From the centre C with any interval let there be deſcribed the circle nom, cutting the right line CP in n, the perimeter of the wheel BP in o, and the curvilinear path AP in m; and from the centre V with the interval Vo let there be deſcribed a circle cutting VP produced in q.

Becauſe the wheel in its progreſs always revolves about the point of contact B, it is manifeſt that the right line BP is perpendicular to that curve line AP which the point P of the wheel deſcribes, and therefore that the right line VP will touch this curve in the point P. Let the radius of the circle nom be gradually increaſed or diminiſhed ſo that at laſt it become equal to the diſtance CP; and by reaſon of the ſimilitude of the evaneſcent figure $Pnomq$, and the figure $PFGVI$, the ultimate ratio of the evaneſcent lineolæ Pm, Pn, Po, Pq, that is, the ratio of the momentary mutations of the curve AP, the right line CP, the circular arc BP, and the right line VP, will be the ſame as of the lines PV, PF, PG, PI, reſpectively. But ſince VF is perpendicular to CF, and VH to CV, and therefore the angles HVG, VCF equal; and the angle VHG (becauſe the angles of the quadrilateral figure $HVEP$ are right in V and P) is equal to the angle CEP, the triangles VHG, CEP will be ſimilar; and thence it will come to paſs that as EP is to CE ſo is HG to HV or HP, and ſo KI to KP, and by compoſition or diviſion as CB to CE ſo is PI to PK, and doubling the conſequents as CB to $2CE$ ſo PI to PV, and ſo is Pq to Pm. Therefore the decrement of the line VP, that is the increment of the line $BV-VP$ to the increment of the curve line AP is in a given ratio of CB to $2CE$, and therefore (by cor. lem. 4.) the lengths $BV-VP$ and AP generated by thoſe increments, are in the ſame ratio. But if BV be radius, VP is the coſine of the angle BVP or $\frac{1}{2}BEP$, and therefore $BV-VP$ is the verſed ſine of the ſame angle; and therefore in this wheel whoſe radius is $\frac{1}{2}BV$, $BV-VP$ will be double the verſed ſine of the arc $\frac{1}{2}BP$. Therefore AP is to double the verſed ſine of the arc $\frac{1}{2}BP$ as $2CE$ to CB. *Q. E. D.*

The line AP in the former of theſe propoſitions we ſhall name the cycloid without the globe, the other in the latter propoſition the cycloid within the globe, for diſtinction ſake.

Cor. 1. Hence if there be deſcribed the entire cycloid ASL and the ſame be biſected in S, the length of the part PS will be to the length PV (which is the double of the ſine of the angle

VBP, when *EB* is radius) as 2*CE* to *CB*, and therefore in a given ratio.

COR. 2. And the length of the ſemi-perimeter of the cycloid *AS* will be equal to a right line which is to the diameter of the wheel *BV* as 2*CE* to *CB*.

PROPOSITION L. PROBLEM XXXIII.

To cauſe a pendulous body to oſcillate in a given cycloid.

Let there be given within the globe *QVS*, (*Pl.* 19. *Fig.* 3.) deſcribed with the centre *C*, the cycloid *QRS*, biſected in *R*, and meeting the ſuperficies of the globe with its extreme points *Q* and *S* on either hand. Let there be drawn *CR* biſecting the arc *QS* in *O*, and let it be produced to *A* in ſuch ſort that *CA* may be to *CO* as *CO* to *CR*. About the centre *C*, with the interval *CA*, let there be deſcribed an exterior globe *DAF*, and within this globe; by a wheel whoſe diameter is *AO*, let there be deſcribed two ſemi-cycloids *AQ*, *AS*, touching the interior globe in *Q* and *S*, and meeting the exterior globe in *A*. From that point *A*, with a thread *APT* in length equal to the line *AR*, let the body *T* depend, and oſcillate in ſuch manner between the two ſemi-cycloids *AQ*, *AS* that as often as the pendulum parts from the perpendicular *AR*, the upper part of the thread *AP* may be applied to that ſemi-cycloid *APS* towards which the motion tends, and fold it ſelf round that curve line, as if it were ſome ſolid obſtacle,

ſtacle; the remaining part of the ſame thread *PT* which has not yet touched the ſemi-cycloid continuing ſtraight. Then will the weight *T* oſcillate in the given cycloid *QRS*. *Q. E. F.*

For let the thread *PT* meet the cycloid *QRS* in *T*, and the circle *QOS* in *V*, and let *CV* be drawn; and to the rectilinear part of the thread *PT* from the extreme points *P* and *T* let there be erected the perpendiculars *BP*, *TW*, meeting the right line *CV* in *B* and *W*. It is evident from the conſtruction and generation of the ſimilar figures *AS*, *SR*, that thoſe perpendiculars *PB*, *TW*, cut off from *CV* the lengths *VB*, *VW* equal to the diameters of the wheels *OA*, *OR*. Therefore *TP* is to *VP* (which is double the ſine of the angle *VBP* when $\frac{1}{2}BV$ is radius) as *BW* to *BV*, or $AO+OR$ to *AO*, that is (ſince *CA* and *CO*, *CQ* and *CR*, and by diviſion *AO* and *QR* are proportional) as $CA+CO$ to *CA*, or, if *BV* be biſected in *E*, as $2CE$ to *CB*. Therefore (by cor. 1. prop. 49) the length of the rectilinear part of the thread *PT* is always equal to the arc of the cycloid *PS*, and the whole thread *APT* is always equal to the half of the cycloid *APS*, that is (by cor. 2. prop. 49.) to the length *AR*. And therefore contrarywiſe, if the ſtring remain always equal to the length *AR* the point *T* will always move in the given cycloid *QRS*. *Q. E. D.*

Cor. The ſtring *AR* is equal to the ſemi-cycloid *AS*, and therefore has the ſame ratio to *AC* the ſemi-diameter of the exterior globe as the like ſemi-cycloid *SR* has to *CQ* the ſemi-diameter of the interior globe.

Pro-

PROPOSITION LI. THEOREM XVIII.

If a centripetal force tending on all ſides to the centre C *of a globe* (Pl. 19. Fig. 4.) *be in all places as the diſtance of the place from the centre, and by this force alone acting upon it, the body* T *oſcillate* (*in the manner above deſcribed*) *in the perimeter of the cycloid* QRS; *I ſay, that all the oſcillations how unequal ſoever in themſelves will be performed in equal times.*

For upon the tangent *TW* infinitely produced let fall the perpendicular *CX* and join *CT*. Becauſe the centripetal force with which the body *T* is impelled towards *C* is as the diſtance *CT*, let this (by cor. 2. of the laws) be reſolved into the parts *CX*, *TX* of which *CX* impelling the body directly from *P* ſtretches the thread *PT*, and by the reſiſtance the thread makes to it is totally employed, producing no other effect; but the other part *TX*, impelling the body tranſverſely or towards *X*, directly accelerates the motion in the cycloid. Then it is plain that the acceleration of the body, proportional to this accelerating force, will be every moment as the length *TX*, that is, (becauſe *CV*, *WV*, and *TX*, *TW* proportional to them are given) as the length *TW*, that is (by cor. 1. prop. 49.) as the length of the arc of the cycloid *TR*. If therefore two pendulums *APT*, *Apt* be unequally drawn aſide from the perpendicular *AR*, and let fall together, their accelerations will be always as the arcs to be deſcribed *TR*, *tR*. But the parts deſcribed at the beginning of the motion are as the accelerations, that is, as the wholes that are to be deſcribed at the beginning, and therefore the parts which remain to be deſcribed and the ſubſequent accelerations proportional to thoſe parts, are alſo as the wholes, and ſo on. Therefore the accelerations, and conſequently the velocities generated, and the parts deſcribed with thoſe velocities, and the parts to be deſcribed, are always as the wholes; and therefore the parts to be deſcribed preſerving a given ratio to each other will vaniſh together, that is, the two bodies oſcillating will arrive together at the perpendicular *AR*. And ſince on the other hand the aſcent of the pendulums from the loweſt place *R* through the ſame cycloidal arcs with a retrograde motion, is retarded in the ſeveral places they paſs through by the ſame forces by which their deſcent was accelerated, 'tis plain that the velocities of their aſcent and deſcent through the ſame arcs are equal, and conſequently performed in equal times; and therefore ſince the two parts of the cycloid *RS* and *RQ* lying on either ſide of the perpendicular are ſimilar and equal, the two pendulums will perform as well the wholes as the halves of their oſcillations in the ſame times. *Q. E. D.*

Cor. The force with which the body *T* is accelerated or retarded in any place *T* of the cycloid, is to the whole weight of the ſame body in the higheſt place *S* or *Q*, as the arc of the cycloid *TR* is to the arc *SR* or *QR*.

Fig. 1. p. 200.

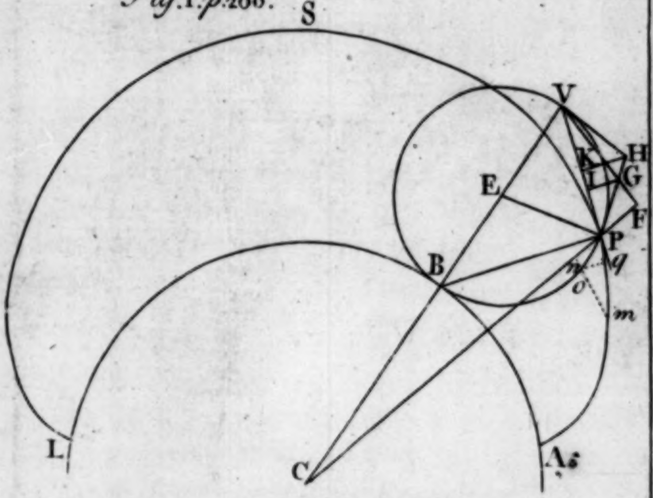


Fig. 2. ib.

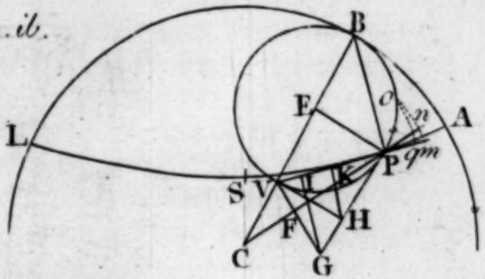


Fig. 3. p. 203.

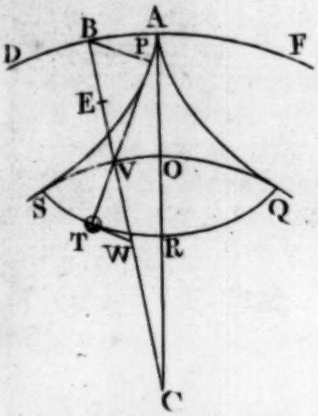
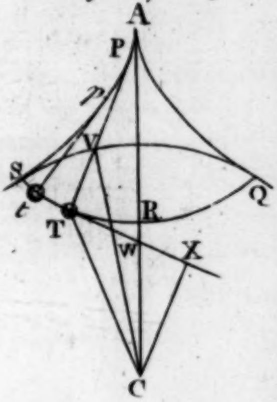


Fig. 4. p. 205.



PROPOSITION LII. PROBLEM XXXIV.

To define the velocities of the pendulums in the ſeveral places, and the times in which both the entire oſcillations, and the ſeveral parts of them are performed.

About any centre *G* (*Pl.* 20. *Fig.* 1.) with the interval *GH* equal to the arc of the cycloid *RS*, deſcribe a ſemi-circle *HKM* biſected by the ſemidiameter *GK*. And if a centripetal force proportional to the diſtance of the places from the centre tend to the centre *G*, and it be in the perimeter *HIK* equal to the centripetal force in the perimeter of the globe *QOS* tending towards its centre, and at the ſame time that the pendulum *T* is let fall from the higheſt place *S*, a body as *L* is let fall from *H* to *G*; then becauſe the forces which act upon the bodies are equal at the beginning, and always proportional to the ſpaces to be deſcribed *TR*, *LG*, and therefore if *TR* and *LG* are equal, are alſo equal in the places *T* and *L*, it is plain that thoſe bodies deſcribe at the beginning equal ſpaces *ST*, *HL*, and therefore are ſtill acted upon equally, and continue to deſcribe equal ſpaces. Therefore by prop. 38. the time in which the body deſcribes the arc *ST* is to the time of one oſcillation, as the arc *HI* the time in which the body *H* arrives at *L*, to the ſemi-periphery *HKM*, the time in which the body *H* will come to *M*. And the velocity of the pendulous body in the place *T* is to its velocity in the loweſt place *R*, that is, the velocity of the body *H* in the place *L* to its velocity in the place *G*, or the momentary increment of the line *HL* to the momentary increment of the line *HG*, (the arcs *HI*, *HK* increaſing with an equable flux) as the ordinate *LI* to the radius *GK*, or as $\sqrt{SR^2 - TR^2}$ to *SR*. Hence ſince in unequal oſcillations there are deſcribed in equal times arcs proportional to the entire arcs of the oſcillations; there are obtained from the times given, both the velocities and the arcs deſcribed in all the oſcillations univerſally. Which was firſt required.

Let now any pendulous bodies oſcillate in different cycloids deſcribed within different globes, whoſe abſolute forces are alſo different; and if the abſolute force of any globe *QOS* be called V, the accelerative force with which the pendulum is acted on in the circumference of this globe, when it begins to move directly towards its centre, will be as the diſtance of the pendulous body from that centre and the abſolute force of the globe conjunctly, that is, as $CO \times V$. Therefore the lineola *HY* which is as this accelerative force $CO \times V$ will be deſcribed in a given time; and if there be erected the perpendicular *YZ* meeting the circumference in *Z*, the naſcent arc *HZ* will denote that given time. But that naſcent arc *HZ* is in the ſubduplicate ratio of the rectangle *GHY*, and therefore as $\sqrt{GH \times CO \times V}$. Whence the time of an entire oſcillation in the cycloid *QRS* (it being as the ſemi-periphery HKM which denotes that entire oſcillation, directly; and as the arc HZ which in like manner denotes a given time inverſely) will be as GH directly and $\sqrt{GH \times CO \times V}$ inverſely, that is, becauſe GH and SR are equal, as $\sqrt{\frac{SR}{CO \times V}}$, or (by cor. prop. 50.) as $\sqrt{\frac{AR}{AC \times V}}$. Therefore the oſcillations in all globes and cycloids, performed with what abſolute forces ſoever, are in a ratio compounded of the ſubduplicate ratio of the length of the ſtring directly, and the ſubduplicate ratio of the diſtance between the point of ſuſpenſion and the centre of the globe inverſely, and the ſubduplicate ratio of the abſolute force of the globe inverſely alſo. *Q. E. I.*

COR. 1. Hence alſo the times of oſcillating, falling, and revolving bodies may be compared among themſelves. For if the diameter of the wheel with which the cycloid is deſcribed within the globe is ſuppoſed equal to the ſemi-diameter of the globe, the cycloid will become a right line paſſing through the centre of the globe, and the oſcillation will be changed into a deſcent and ſubſequent aſcent in that right line. Whence there is given both the time of the deſcent from any place to the centre, and the time equal to it in which the body revolving uniformly about the centre of the globe at any diſtance deſcribes an arc of a quadrant. For this time (by caſe 2.) is to the time of half the oſcillation in any cycloid QRS as 1 to $\sqrt{\frac{AR}{AC}}$.

COR. 2. Hence alſo follow what Sir *Chriſtopher Wren* and M. *Huygens* have diſcovered concerning the vulgar cycloid. For if the diameter of the globe be infinitely increaſed, its ſphærical ſuperficies will be changed into a plane, and the centripetal force will act uniformly in the direction of lines perpendicular to that plane, and this cycloid of ours will become the ſame with the common cycloid. But in that caſe the length of the arc of the cycloid between that plane and the deſcribing point, will become equal to four times the verſed ſine of half the arc of the wheel between the ſame plane and the deſcribing point as was diſcovered by Sir *Chriſtopher Wren*. And a pendulum between two ſuch cycloids will oſcillate in a ſimilar and equal cycloid in equal times as M. *Huygens* demonſtrated. The deſcent of heavy bodies alſo in the time of one oſcillation will be the ſame as M. *Huygens* exhibited.

The propoſitions here demonſtrated are adapted to the true conſtitution of the Earth, in ſo far as wheels moving in any of its great circles will deſcribe by the motions of nails fixed in their perimeters, cycloids without the globe; and pendulums in mines and deep caverns of the Earth muſt oſcillate in cycloids within the globe, that thoſe oſcillations may be performed in equal times. For gravity (as will be ſhewn in the third book) decreaſes in its progreſs from the ſuperficies of the Earth; upwards in a duplicate ratio of the diſtances from the centre of the earth; downwards in a ſimple ratio of the ſame.

Proposition LIII. Problem XXXV.

Granting the quadratures of curvilinear figures, it is required to find the forces with which bodies moving in given curve lines may always perform their oſcillations in equal times.

Let the body T (*Pl.* 20. *Fig.* 2.) oſcillate in any curve line $STRQ$, whoſe axis is AR paſſing through the centre of force C. Draw TX touching that curve in any place of the body T, and in that tangent TX take TY equal to the arc TR. The length of that arc is known from the common methods uſed for the quadratures of figures. From the point Y draw the right line YZ perpendicular to the tangent. Draw CT meeting that perpendicular in Z, and the centripetal force will be proportional to the right line TZ. *Q. E. I.*

For if the force with which the body is attracted from T towards C be expreſſed by the right line TZ taken proportional to it, that force will be reſolved into two forces TY, YZ, of which YZ drawing the body in the direction of the length of the thread PT, does not at all change its motion; whereas the other force TY directly accelerates or retards its motion in the curve $STRQ$. Wherefore ſince that force is as the ſpace to be deſcribed TR, the accelerations or retardations of the body in deſcribing two proportional parts (a greater and a leſs) of two oſcillations, will be always as thoſe parts, and therefore will cauſe thoſe parts to be deſcribed together. But bodies which continually deſcribe together parts proportional to the wholes, will deſcribe the wholes together alſo. *Q. E. D.*

Cor. 1. Hence if the body *T* (*Pl.* 20. *Fig.* 3.) hanging by a rectilinear thread *AT* from the centre *A*, deſcribe the circular arc *STRQ*, and in the mean time be acted on by any force tending downwards with parallel directions, which is to the uniform force of gravity as the arc *TR* to its ſine *TN*, the times of the ſeveral oſcillations will be equal. For becauſe *TZ*, *AR* are parallel, the triangles *ATN*, *ZTT* are ſimilar; and therefore *TZ* will be to *AT* as *TT* to *TN*; that is, if the uniform force of gravity be expreſſed by the given length *AT* the force *TZ* by which the oſcillations become iſochronous, will be to the force of gravity *AT*, as the arc *TR* equal to *TT* is to *TN* the ſine of that arc.

Cor. 2. And therefore in clocks, if forces were impreſſed by ſome machine upon the pendulum which preſerves the motion, and ſo compounded with the force of gravity, that the whole force tending downwards ſhould be always as a line produced by applying the rectangle under the arc *TR* and the radius *AR* to the ſine *TN*, all the oſcillations will become iſochronous.

Proposition LIV. Problem XXXVI.

Granting the quadratures of curvilinear figures, it is required to find the times, in which bodies by means of any centripetal force will deſcend or aſcend in any curve lines deſcribed in a plane paſſing through the centre of force.

Let the body deſcend from any place *S* (*Pl.* 20. *Fig.* 4.) and move in any curve *STtR* given in a plane paſſing through the centre of force *C*. Join *CS*, and let it be divided into innumerable equal parts, and let *Dd* be one of thoſe parts. From the centre *C*, with the intervals *CD*, *Cd*, let the circles *DT*, *dt* be deſcribed, meeting the curve line *STtR* in *T* and *t*. And becauſe the law of centripetal force is given, and alſo the altitude *CS* from which the body at firſt fell; there will be given the velocity of the body in any other altitude *CT* (by prop. 39.) But the time in which the body deſcribes the lineola *Tt* is as the length of that lineola, that is, as the ſecant of the angle *tTC* directly, and the velocity inverſely. Let the ordinate *DN*, proportional to this time, be made perpendicular to the right line *CS* at the point *D*, and becauſe *Dd* is given, the rectangle $Dd \times DN$ that is, the area *DNnd*, will be proportional to the ſame time. Therefore if *PNn* be a curve line in which the point *N* is perpetually found, and its aſymptote be the right line *SQ* ſtanding upon the line *CS* at right angles, the area *SQPND* will be proportional to the time in which the body in its deſcent hath deſcribed the line *ST*; and therefore that area being found the time is alſo given. *Q. E. I.*

PROPOSITION LV. THEOREM XIX.

If a body move in any curve ſuperficies whoſe axis paſſes through the centre of force, and from the body a perpendicular be let fall upon the axis; and a line parallel and equal thereto be drawn from any given point of the axis; I ſay, that this parallel line will deſcribe an area proportional to the time.

Let *BKL* (*Pl.* 20. *Fig.* 5.) be a curve ſuperficies, *T* a body revolving in it, *STR* a trajectory which the body deſcribes in the ſame, *S* the beginning of the trajectory, *OMK* the axis of the curve ſuperficies, *TN* a right line let fall perpendicularly from the body to the axis; *OP* a line parallel and equal thereto drawn from the given point *O* in the axis; *AP* the orthographic projection of the trajectory deſcribed by the point *P* in the plane *AOP* in which the revolving line *OP* is found; *A* the beginning of that projection anſwering to the point *S*; *TC* a right line drawn from the body to the centre; *TG* a part thereof proportional to the centripetal force with which the body tends towards the centre *C*; *TM* a right line perpendicular to the curve ſuperficies; *TI* a part thereof proportional to the force of preſſure with which the body urges the ſuperficies, and therefore with which it is again repelled by the ſuperficies towards *M*; *PTF* a right line parallel to the axis and paſſing through the body, and *GF*, *IH* right lines let fall perpendicularly from the points *G* and *I* upon that parallel *PHTF*. I ſay now that the area *AOP*, deſcribed by the radius *OP* from the beginning of the motion, is proportional to the time. For the force *TG* (by cor. 2. of the laws of motion) is reſolved into the forces *TF*, *FG*; and the force *TI* into the forces *TH*, *HI*; but the forces *TF*, *TH*, acting in the direction of the line *PF* perpendicular to the plane *AOP*, introduce no change in the motion of the body but in a direction perpendicular to that plane. Therefore its motion ſo far as it has the ſame direction with the poſition of the plane, that is, the motion of the point *P*, by which the projection *AP* of the trajectory is deſcribed in that plane, is the ſame as if the forces *TF*, *TH* were taken away, and the body were acted on by the forces *FG*, *HI* alone; that is, the ſame as if the body were to deſcribe in the plane *AOP* the curve *AP* by means of a centripetal force tending to the centre *O*, and equal to the ſum of the forces *FG* and *HI*. But with ſuch a force as that (by prop. 1.) the area *AOP* will be deſcribed proportional to the time. *Q. E. D.*

COR. By the ſame reaſoning if a body, acted on by forces tending to two or more centres in any the ſame right line *CO*, ſhould deſcribe in a free ſpace any curve line *ST*; the area *AOP* would be always proportional to the time.

PROPOSITION LVI. PROBLEM XXXVII.

Granting the quadratures of curvilinear figures and ſuppoſing that there are given both the law of centripetal force tending to a given centre, and the curve ſuperficies whoſe axis paſſes through that centre; it is required to find the trajectory which a body will deſcribe in that ſuperficies, when going off from a given place with a given velocity, and in a given direction in that ſuperficies.

The laſt conſtruction remaining, let the body *T* go from the given place *S* (*Pl.* 20. *Fig.* 6.) in the direction of a line given by poſition, and turn into the trajectory ſought *STR* whoſe orthographic projection in the plane *BLO* is *AP*. And from the given velocity of the body in the altitude *SC*, its velocity in any other altitude *TC* will be alſo given. With that velocity in a given moment of time let the body deſcribe the particle *Tt* of its trajectory, and let *Pp* be the projection of that particle deſcribed in the plane *AOP*. Join *Op*, and a little circle being deſcribed upon the curve ſuperficies about the centre *T* with the interval *Tt*, let the projection of that little circle in the plane

AOP be the ellipſis *pQ*. And becauſe the magnitude of that little circle *Tt*, and *TN* or *PO* its diſtance from the axis *CO* is alſo given, the ellipſis *pQ* will be given both in kind and magnitude, as alſo its poſition to the right line *PO*. And ſince the area *POp* is proportional to the time, and therefore given becauſe the time is given the angle *POp* will, be given. And thence will be given *p* the common interſection of the ellipſis and the right line *Op*, together with the angle *OPp* in which the projection *APp* of the trajectory cuts the line *OP*. But from thence (by conferring prop. 41. with its 2d cor.) the manner of determining the curve *APp* eaſily appears. Then from the ſeveral points *P* of that projection erecting to the plane *AOP* the perpendiculars *PT* meeting the curve ſuperficies in *T*, there will be given the ſeveral points *T* of the trajectory. *Q. E. I.*

SECTION XI.

Of the motions of bodies tending to each other with centripetal forces.

I have hitherto been treating of the attractions of bodies towards an immoveable centre; tho' very probably there is no ſuch thing exiſtent in nature. For attractions are made towards bodies; and the actions of the bodies attracted and attracting, are always reciprocal and equal by law 3. ſo that if there are two bodies, neither the attracted nor the attracting body is truly at reſt, but both (by cor. 4. of the laws of motion) being as it were mutually attracted, revolve about a common centre of gravity. And if there be more bodies, which are either attracted by one ſingle one which is attracted by them again, or which, all of them, attract each other mutually; theſe bodies will be ſo moved among themſelves, as that their common centre of gravity will either be at reſt, or move uniformly forward in a right line. I ſhall therefore at preſent go on to tieat of the motion of bodies mutually attracting each other; conſidering the centripetal forces as attractions; though perhaps in a phyſical ſtrictneſs they may more truly be called impulſes. But theſe propoſitions are to be conſidered as purely mathematical; and therefore laying aſide all phyſical conſiderations, I make uſe of a familiar way of ſpeaking, to make my ſelf the more eaſily underſtood by a mathematical reader.

PROPOSITION LVII. THEOREM XX.

Two bodies attracting each other mutually, deſcribe ſimilar figures about their common centre of gravity, and about each other mutually.

For the diſtances of the bodies from their common centre of gravity are reciprocally as the bodies; and therefore in a given ratio to each other; and thence by compoſition of ratio's, in a given ratio to the whole diſtance between the bodies. Now theſe diſtances revolve about their common term with an equable angular motion, becauſe lying in the ſame right line they never change their inclination to each other mutually. But right lines that are in a given ratio to each other, and revolve about their terms with an equal angular motion, deſcribe upon planes, which either reſt with thoſe terms, or move with any motion not angular, figures entirely ſimilar round thoſe terms. Therefore the figures deſcribed by the revolution of theſe diſtances are ſimilar. *Q. E. D.*

PROP-

PROPOSITION LVIII. THEOREM XXI.

If two bodies attract each other mutually with forces of any kind, and in the mean time revolve about the common centre of gravity; I ſay that by the ſame forces there may be deſcribed round either body unmoved, a figure ſimilar and equal to the figures which the bodies ſo moving deſcribe round each other mutually.

Let the bodies *S* and *P* (*Pl.* 20. *Fig.* 7.) revolve about their common centre of gravity *C*, proceeding from *S* to *T*, and from *P* to *Q*. From the given point *s*, let there be continually drawn *sp*, *sq*, equal and parallel to *SP*, *TQ*; and the curve *pqv*, which the point *p* deſcribes in its revolution round the immoveable point *s*, will be ſimilar and equal to the curves, which the bodies *S* and *P* deſcribe about each other mutually; and therefore by theor. 20. ſimilar to the curves *ST* and *PQV* which the ſame bodies deſcribe about their common centre of gravity *C*; and that becauſe the proportions of the lines *SC*, *CP*, and *SP* or *sp*, to each other, are given.

CASE 1. The common centre of gravity *C* (by cor. 4. of the laws of motion) is either at reſt, or moves uniformly in a right line. Let us firſt ſuppoſe it at reſt, and in *s* and *p* let there be placed two bodies, one immoveable in *s*, the other moveable in *p*, ſimilar and equal to the bodies *S* and *P*. Then let the right lines *PR* and *pr* touch the curves *PQ* and *pq* in *P* and *p*, and produce *CQ* and *sq* to *R* and *r*. And becauſe the figures *CPRQ*, *sprq* are ſimilar, *RQ* will be to *rq* as *CP* to *sp*, and therefore in a given ratio. Hence if the force with which the body *P* is attracted towards the body *S*, and by conſequence towards the intermediate point the centre *C*, were to the force with which the body *p* is attracted towards the centre *s*, in the ſame given ratio; theſe forces would in equal times attract the bodies from the tangents *PR*, *pr* to the arcs *PQ*, *pq*, through the intervals proportional to them *RQ*, *rq*; and therefore this laſt force (tending to *s*) would make the body *p* revolve in the curve *pqv*, which would become ſimilar to the curve *PQV*, in which the firſt force obliges the body *P* to revolve; and their revolutions would be compleated in the ſame times. But becauſe thoſe forces are not to each other in the ratio of *CP* to *sp*, but (by reaſon of the ſimilarity and equality of the bodies *S* and *s*, *P* and *p*, and the equality of the diſtances *SP*, *sp*) mutually equal; the bodies in equal times will be equally drawn from the tangents; and therefore that the body *p* may be attracted through the greater interval *rq*, there is required a greater time, which will be in the ſubduplicate ratio of the intervals; becauſe by lemma 10. the ſpaces deſcribed at the very beginning of the motion are in a duplicate ratio of the times. Suppoſe then the velocity of the body *p* to be to the velocity of the body *P* in a ſubduplicate ratio of the diſtance *sp* to the diſtance *CP*, ſo that the arcs *pq*, *PQ*, which are in a ſimple proportion to each other, may be deſcribed in times that are in a ſubduplicate ratio of the diſtances; and the bodies *P*, *p*, always attracted by equal forces will deſcribe round the quieſcent centres *C* and *s* ſimilar figures *PQV*, *pqv*, the latter of which *pqv* is ſimilar and equal to the figure which the body *P* deſcribes round the moveable body *S*. *Q. E. D.*

Case 2. Suppoſe now that the common centre of gravity together with the ſpace in which the bodies are moved among themſelves, proceeds uniformly in a right line; and (by cor. 6. of the laws of motion) all the motions in this ſpace will be performed in the ſame manner as before; and therefore the bodies will deſcribe mutually about each other the ſame figures as before, which will be therefore ſimilar and equal to the figure *pqv*. *Q. E. D.*

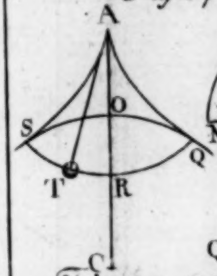
Cor. 1. Hence two bodies attracting each other with forces proportional to their diſtance, deſcribe (by prop. 10.) both round their common centre of gravity, and round each other mutually, concentrical ellipſes; and *vice versâ* if ſuch figures are deſcribed, the forces are proportional to the diſtances.

Cor. 2. And two bodies, whoſe forces are reciprocally proportional to the ſquare of their diſtance, deſcribe, (by prop. 11, 12, 13.) both round their common centre of gravity and round each other mutually, conic ſections having their focus in the centre about which the figures are deſcribed. And *vice versâ*, if ſuch figures are deſcribed, the centripetal forces are reciprocally proportional to the ſquares of the diſtance.

Cor. 3. Any two bodies revolving round their common centre of gravity, deſcribe areas proportional to the

Plate XX. Vol. I. P. 222.

Fig. 1. p. 207.



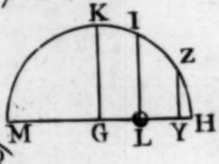


Fig. 2. p. 211.

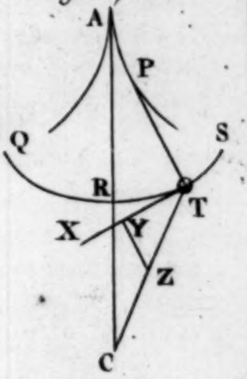


Fig. 3. p. 212.

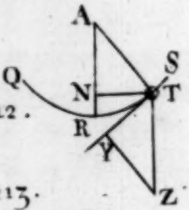


Fig. 4. p. 213.

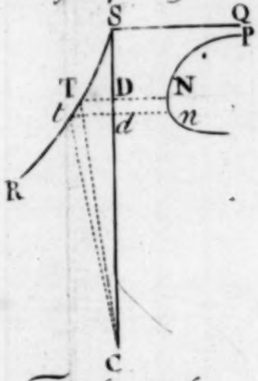


Fig. 5. p. 214.

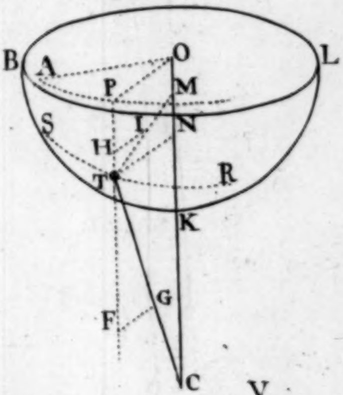


Fig. 6. p. 216.

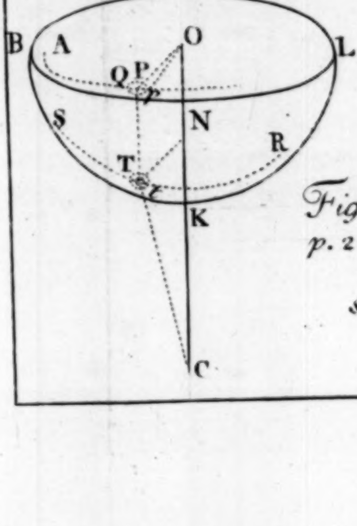
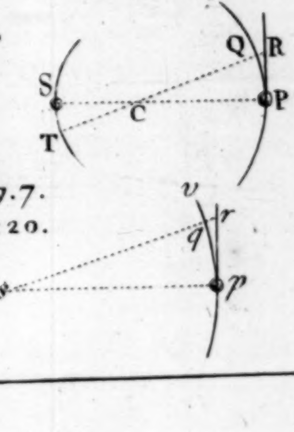


Fig. 7. p. 220.



the times, by radij drawn both to that centre and to other mutually.

PROPOSITION LIX. THEOREM XXII.

The periodic time of two bodies S *and* P *revolving round their common centre of gravity* C, *is to the periodic time of one of the bodies* P *revolving round the other* S *remaining unmoved, and deſcribing a figure ſimilar and equal to thoſe which the bodies deſcribe about each other mutually, in a ſubduplicate ratio of the other body* S *to the ſum of the bodies* S+P.

For by the demonſtration of the laſt propoſition, the times in which any ſimilar arcs *PQ* and *pq* are deſcribed, are in a ſubduplicate ratio of the diſtances *CP* and *SP* or *sp*, that is in a ſubduplicate ratio of the body *S* to the ſum of the bodies *S*+*P*. And by compoſition of ratio's, the ſums of the times in which all the ſimilar arcs *PQ* and *pq* are deſcribed, that is, the whole times in which the whole ſimilar figures are deſcribed, are in the ſame ſubduplicate ratio. *Q. E. D.*

PROPOSITION LX. THEOREM XXIII.

If two bodies S *and* P, *attracting each other with forces reciprocally proportional to the ſquares of their diſtance, revolve about their common centre of gravity; I ſay that the principal axis of the ellipſis which either of the bodies as* P *deſcribes by this motion about the other* S, *will be to the principal axis of the ellipſis, which the ſame body* P *may deſcribe in the ſame periodical time about the other body* S *quieſcent, as the ſum of the two bodies* S+P *to the firſt of two mean proportionals between that ſum and the other body* S.

For if the ellipſes deſcribed were equal to each other, their periodic times by the laſt theorem would be in a ſubduplicate ratio of the body S to the ſum of the bodies $S+P$. Let the periodic time in the latter ellipſis be diminiſhed in that ratio, and the periodic times will become equal; but by prop. 15. the principal axis of the ellipſis, will be diminiſhed in a ratio ſeſquiplicate to the former ratio; that is in a ratio, to which the ratio of S to $S+P$ is triplicate; and therefore that axis will be to the principal axis of the other ellipſis, as the firſt of two mean proportionals between

$S+P$ and S to $S+P$. And inverſely the principal axis of the ellipſis deſcribed about the moveable body, will be to the principal axis of that deſcribed round the immoveable, as $S+P$ to the firſt of two mean proportionals between $S+P$ and S. *Q. E. D.*

PROPOSITION LXI. THEOREM XXIV.

If two bodies attracting each other with any kind of forces, and not otherwiſe agitated or obſtructed, are moved in any manner whatſoever; thoſe motions will be the ſame, as if they did not at all attract each other mutually, but were both attracted with the ſame forces by a third body placed in their common centre of gravity; and the law of the attracting forces will be the ſame in reſpect of the diſtance of the bodies from the common centre, as in reſpect of the diſtance between the two bodies.

For thoſe forces with which the bodies attract each other mutually, by tending to the bodies tend alſo to the common centre of gravity lying directly between them; and therefore are the ſame as if they proceeded from an intermediate body. *Q. E. D.*

And becauſe there is given the ratio of the diſtance of either body from that common centre to the diſtance between the two bodies, there is given of courſe the ratio of any power of one diſtance to the ſame power of the other diſtance; and alſo the ratio of any quantity derived in any manner from one of the diſtances compounded any how with given quantities, to another quantity, derived in like manner from the other diſtance, and as many given quantities having that given ratio of the diſtances to the firſt. Therefore if the force with which one body is attracted by another be directly or inverſely as the diſtance of the bodies from each other, or as any power of that diſtance; or laſtly as any quantity derived after any manner from that diſtance compounded with given quantities; then will the ſame force with which the ſame body is attracted to the common centre of gravity, be in like manner directly or inverſely as the diſtance of the attracted body from the common centre, or as any power of that diſtance, or laſtly as a quantity derived in like ſort from that diſtance compounded with analogous given quantities. That is, the law of attracting force will be the ſame with reſpect to both diſtances. *Q. E. D.*

PROPOSITION LXII. PROBLEM XXXVIII.

To determine the motions of two bodies which attract each other with forces reciprocally proportional to the ſquares of the diſtance between them, and are let fall from given places.

The bodies, by the laſt theorem, will be moved in the ſame manner as if they were attracted by a third placed in the common centre of their gravity; and by the hypotheſis that centre will be quieſcent at the beginning of their motion, and therefore (by cor. 4. of the laws of motion) will be always quieſcent. The motions of the bodies are therefore to be determined (by prob. 25.) in the ſame manner as if they were impelled by forces tending to that centre; and then we ſhall have the motions of the bodies attracting each other mutually. *Q. E. I.*

PROPOSITION LXIII. PROBLEM XXXIX.

To determine the motions of two bodies attracting each other with forces reciprocally proportional to the ſquares of their diſtance, and going off from given places in given directions, with given velocities.

The motions of the bodies at the beginning being given, there is given alſo the uniform motion of the common centre of gravity, and the motion of the ſpace which moves along with this centre uniformly in a right line, and alſo the very firſt, or beginning motions of the bodies in reſpect of this ſpace. Then (by cor. 5. of the laws, and the laſt theorem) the ſubſequent motions will be performed in the ſame manner in that ſpace, as if that ſpace together with the common centre of gravity were at reſt, and as if the bodies did not attract each other, but were attracted by a third body placed in that centre. The motion therefore in this moveable ſpace of each body going off from a given place, in a given direction, with a given velocity, and acted upon by a centripetal force tending to that centre, is to be determined by prob. 9. and 26. and at the ſame time will be obtained the motion of the other round the ſame centre. With this motion compound the uniform progreſſive motion of the entire ſyſtem of the ſpace and the bodies revolving in it, and there will be obtained the abſolute motion of the bodies in immoveable ſpace. *Q. E. I.*

PROPOSITION LXIV. PROBLEM XL.

Suppoſing forces with which bodies mutually attract each other to increaſe in a ſimple ratio of their diſtances from the centres; it is required to find the motions of ſeveral bodies among themſelves.

Suppoſe the two firſt bodies *T* and *L* (*Pl.* 21. *Fig.* 1.) to have their common centre of gravity in *D*. Theſe by cor. 1. theor. 21. will deſcribe ellipſes having their centres in *D*, the magnitudes of which ellipſes are known by prob. 5.

Let now a third body *S* attract the two former *T* and *L* with the accelerative forces *ST*, *SL*, and let it be attracted again by them. The force *ST* (by cor. 2. of the laws of motion) is reſolved into the forces *SD*, *DT*; and the force *SL* into the forces *SD* and *DL*. Now the forces *DT*, *DL*, which are as their ſum *TL*, and therefore as the accelerative forces with which the bodies *T* and *L* attract each other mutually, added to the forces of the bodies *T* and *L*, the firſt to the firſt, and the laſt to the laſt, compoſe forces proportional to the diſtances *DT* and *DL* as before, but only greater than thoſe former forces; and therefore (by cor. 1. prop. 10. and cor. 1. and 8. prop. 4.) they will cauſe thoſe bodies to deſcribe ellipſes as before, but with a ſwifter motion. The remaining accelerative forces *SD* and *SD*, by the motive forces, $SD \times T$ and $SD \times L$ which are as the bodies, attracting thoſe bodies equally, and in the direction of the lines *TI*, *LK* parallel to *DS*, do not at all change their ſituations with reſpect to one another, but cauſe them equally to approach to the line *IK*; which muſt be imagined drawn through the middle of the body *S*, and perpendicular to the line *DS*. But that approach to the line *IK* will be hindered by cauſing the ſyſtem of the bodies *T* and *L* on one ſide, and the body *S* on the other with proper velocities to revolve round the common centre of gravity *C*. With ſuch a motion the body *S*, becauſe the ſum of the motive forces $SD \times T$ and $SD \times L$ is proportional to the diſtance *CS*, tends to the centre *C*, will deſcribe an ellipſis round the ſame centre *C*; and the point *D*, becauſe the lines *CS* and *CD* are proportional, will deſcribe a like ellipſis over-againſt it. But the bodies *T* and *L*, attracted by the motive forces $SD \times T$ and $SD \times L$, the firſt by the firſt, and the laſt by the laſt, equally and in the direction of the parallel lines *TI* and *LK* as was ſaid before, will (by cor. 5. and 6. of the laws of motion) continue to deſcribe their ellipſes round the moveable centre *D* as before. *Q. E. I.*

Let there be added a fourth body *V*, and by the like reaſoning it will be demonſtrated that this body and the point *C* will deſcribe ellipſes about the common centre of gravity *B*; the motions of the bodies *T*, *L*, and *S* round the centres *D* and *C* remaining the ſame as before; but accelerated. And by the ſame method one may add yet more bodies at pleaſure. *Q. E. I.*

This would be the caſe, though the bodies *T* and *L* attract each other mutually with accelerative forces either greater or leſs than thoſe with which they attract the other bodies in proportion to their diſtance. Let all the mutual accelerative attractions be to each other as the diſtances multiplyed into the attracting bodies; and from what has gone before it will eaſily be concluded that all the bodies will deſcribe different ellipſes with equal periodical times about their common centre of gravity *B*, in an immoveable plane. *Q. E. I.*

PRO-

PROPOSITION LXV. THEOREM XXV.

Bodies, whoſe forces decreaſe in a duplicate ratio of their diſtances from their centres, may move among themſelves in ellipſes; and by radij drawn to the foci may deſcribe area's proportional to the times very nearly.

In the laſt propoſition we demonſtrated that caſe in which the motions will be performed exactly in ellipſes. The more diſtant the law of the forces is from the law in that caſe, the more will the bodies diſturb each others motions; neither is it poſſible that bodies attracting each other mutually according to the law ſuppoſed in this propoſition ſhould move exactly in ellipſes unleſs by keeping a certain proportion of diſtances from each other. However in the following caſes the orbits will not much differ from ellipſes.

CASE 1. Imagine ſeveral leſſer bodies to revolve about ſome very great one at different diſtances from it, and ſuppoſe abſolute forces tending to every one of the bodies, proportional to each. And becauſe (by cor. 4. of the laws) the common centre of gravity of them all is either at reſt or moves uniformly forward in a right line, ſuppoſe the leſſer bodies ſo ſmall that the great body may be never at a ſenſible diſtance from that centre; and then the great body will, without any ſenſible error, be either at reſt or move uniformly forward in a right line; and the leſſer will revolve about that great one in ellipſes, and by radij drawn thereto will deſcribe areas proportional to the times; if we except the errors that may be introduced by the receding of the great body from the common centre of gravity, or by the mutual actions of the leſſer bodies upon each other. But the leſſer bodies may be ſo far diminiſhed, as that this receſs and the mutual actions of the bodies on each other may become leſs than any aſſignable; and therefore ſo as that the orbits may become ellipſes, and the areas anſwer to the times, without any error that is not leſs than any aſſignable. *Q. E. O.*

Case 2. Let us imagine a ſyſtem of leſſer bodies revolving about a very great one in the manner juſt deſcribed, or any other ſyſtem of two bodies revolving about each other to be moving uniformly forward in a right line, and in the mean time to be impelled ſide-ways by the force of another vaſtly greater body ſituate at a great diſtance. And becauſe the equal accelerative forces with which the bodies are impelled in parallel directions do not change the ſituation of the bodies with reſpect to each other, but only oblige the whole ſyſtem to change its place while the parts ſtill retain their motions among themſelves; it is manifeſt, that no change in thoſe motions of the attracted bodies can ariſe from their attractions towards the greater, unleſs by the inequality of the accelerative attractions, or by the inclinations of the lines towards each other, in whoſe directions the attractions are made. Suppoſe therefore all the accelerative attractions made towards the great body to be among themſelves as the ſquares of the diſtances reciprocally; and then, by increaſing the diſtance of the great body till the differences of the right lines drawn from that to the others in reſpect of their length, and the inclinations of thoſe lines to each other, be leſs than any given, the motions of the parts of the ſyſtem will continue without errors that are not leſs than any given. And becauſe by the ſmall diſtance of thoſe parts from each other, the whole ſyſtem is attracted as if it were but one body, it will therefore be moved by this attraction as if it were one body; that is, its centre of gravity will deſcribe about the great body one of the conic ſections (that is, a parabola or hyperbola when the attraction is but languid, and an ellipſis when it is more vigorous) and by radij drawn thereto it will deſcribe area's proportional to the times, without any errors but thoſe which ariſe from the diſtances of the parts, which are by the ſuppoſition exceeding ſmall, and may be diminiſhed at pleaſure. *Q. E. O.*

By a like reaſoning one may proceed to more compounded caſes ***in infinitum.***

Cor. 1. In the ſecond caſe, the nearer the very great body approaches to the ſyſtem of two or more revolving bodies, the greater will the perturbation be of the motions of the parts of the ſyſtem among themſelves; becauſe the inclinations of the lines drawn from that great body to thoſe parts become greater; and the inequality of the proportion is alſo greater.

Cor. 2. But the perturbation will be greateſt of all, if we ſuppoſe the accelerative attractions of the parts of the ſyſtem towards the greateſt body of all are not to each other reciprocally as the ſquares of the diſtances from that great body; eſpecially if the inequality of this proportion be greater than the inequality of the proportion of the diſtances from the great body. For if the accelerative force, acting in parallel directions and equally, cauſes no perturbation in the motions of the parts of the ſyſtem, it muſt of courſe, when it acts unequally, cauſe a perturbation ſomewhere, which will be greater or leſs as the inequality is greater or leſs. The exceſs of the greater impulſes acting upon ſome bodies, and not acting upon others, muſt neceſſarily change their ſituation among themſelves. And this perturbation, added to the perturbation ariſing from the inequality and inclination of the lines, makes the whole perturbation greater.

Cor. 3. Hence if the parts of this ſyſtem move in ellipſes or circles without any remarkable perturbation; it is manifeſt, that if they are at all impelled by accelerative forces tending to any other bodies, the impulſe is very weak, or elſe is impreſſed very near equally and in parallel directions upon all of them.

PROPOSITION LXVI. THEOREM XXVI.

If three bodies whoſe forces decreaſe in a duplicate ratio of the diſtances, attract each other mutually; and the accelerative attractions of any two towards the third be between themſelves reciprocally as the ſquares of the diſtances; and the two leaſt revolve

volve about the greateſt; I ſay that the interior of the two revolving bodies will, by radij drawn to the innermoſt and greateſt, deſcribe round that body, area's more proportional to the times, and a figure more approaching to that of an ellipſis having its focus in the point of concourſe of the radij, if that great body be agitated by thoſe attractions, than it would do if that great body were not attracted at all by the leſſer, but remained at reſt; or than it would if that great lody were very much more or very much leſs attracted, or very much more or very much leſs agitated by the attractions.

This appears plainly enough from the demonſtration of the ſecond corollary of the foregoing propoſition; but it may be made out after this manner by a way of reaſoning more diſtinct and more univerſally convincing.

CASE 1. Let the leſſer bodies *P* and *S* (*Pl.* 21. *Fig.* 2.) revolve in the ſame plane about the greateſt body *T*, the body *P* deſcribing the interior orbit *PAB*, and *S* the exterior orbit *ESE*. Let *SK* be the mean diſtance of the bodies *P* and *S*; and let the accelerative attraction of the body *P* towards *S*, at that mean diſtance, be expreſſed by that line *SK*. Make *SL* to *SK* as the ſquare of *SK* to the ſquare of *SP*, and *SL* will be the accelerative attraction of the body P towards S at any diſtance SP. Join PT, and draw LM parallel to it meeting ST in M; and the attraction SL will be reſolved (by cor. 2. of the laws of motion) into the attractions SM, LM. And ſo the body P will be urged with a threefold accelerative force. One of theſe forces tends towards T, and ariſes from the mutual attraction of the bodies T and P. By this force alone the body P would deſcribe round the body T, by the radius PT, areas proportional to the times, and an ellipſis whoſe focus is in the centre of the body T; and this it would do whether the body T remained unmoved, or whether it were agitated by that attraction. This appears from prop. 11. and cor. 2 & 3 of theor. 21. The other force is that of the attraction LM, which becauſe it tends from P to T, will be ſuper-added to and coincide with the former force; and cauſe the area's to be ſtill proportional to the times, by cor. 3. theor. 21. But becauſe it is not reciprocally proportional to the ſquare of the diſtance PT, it will compoſe when added to the former, a force varying from that proportion; which variation will be the greater, by how much the proportion of this force to the former is greater, *cæteris paribus*. Therefore ſince by prop. 11. and by cor. 2. theor. 21. the force with which the ellipſis is deſcribed about the focus T ought to be directed to that focus; and to be reciprocally proportional to the ſquare of the diſtance PT; that compounded force varying from that proportion will make the orbit PAB vary from the figure of an ellipſis that has its focus in the point T; and ſo much the more by how much the variation from that proportion is greater;

and by conſequence by how much the proportion of the ſecond force *LM* to the firſt force is greater, *cæteris paribus.* But now the third force *SM,* attracting the body *P* in a direction parallel to *ST,* compoſes with the other forces a new force which is no longer directed from *P* to *T*; and which varies ſo much more from this direction, by how much the proportion of this third force to the other forces is greater *cæteris paribus*; and therefore cauſes the body *P* to deſcribe, by the radius *TP*, area's no longer proportional to the times; and therefore makes the variation from that proportionality ſo much greater by how much the proportion of this force to the others is greater. But this third force will increaſe the variation of the orbit *PAB* from the elliptical figure before mentioned upon two accounts; firſt becauſe that force is not directed from *P* to *T*; and ſecondly becauſe it is not reciprocally proportional to the ſquare of the diſtance *PT.* Theſe things being premiſed, it is manifeſt, that the area's are then moſt nearly proportional to the times, when that third force is the leaſt poſſible, the reſt preſerving their former quantity; and that the orbit *PAB* does then approach neareſt to the elliptical figure above-mentioned, when both the ſecond and third, but eſpecially the third force, is the leaſt poſſible; the firſt force remaining in its former quantity.

Let the accelerative attraction of the body *T* towards *S* be expreſſed by the line *SN*; then if the accelerative attractions *SM* and *SN* were equal, theſe, attracting the bodies *T* and *P* equally and in parallel directions, would not at all change their ſituation with reſpect to each other. The motions of the bodies between themſelves would be the ſame in that caſe as if thoſe attractions did not act at all, by cor. 6. of the laws of motion. And by a like reaſoning if the attraction *SN* is leſs than the attraction *SM*, it will take away out of the attraction *SM* the part *SN*, ſo that there will remain only the part (of the attraction) *MN*, to diſturb the proportionality of the area's and times, and the elliptical figure of the orbit. And in like manner if the attraction *SN* be greater than the attraction *SM*, the perturbation of the orbit and proportion will be produced by the difference *MN* alone. After this manner the attraction *SN* reduces always the attraction *SM* to the attraction *MN*, the firſt and ſecond attractions remaining perfectly unchanged; and therefore the area's and times come then neareſt to proportionality, and the orbit *PAB* to the above-mentioned elliptical figure, when the attraction *MN* is either none, or the leaſt that is poſſible; that is, when the accelerative attractions of the bodies *P* and *T* approach as near as poſſible to equality; that is, when the attraction *SN* is neither none at all, nor leſs than the leaſt of all the attractions *SM*, but is as it were a mean between the greateſt and leaſt of all thoſe attractions *SM*, that is, not much greater nor much leſs than the attraction *SK*. *Q. E. D.*

Case. 2. Let now the leſſer bodies *P*, *S*, revolve about a greater *T* in different planes; and the force *LM*, acting in the direction of the line *PT* ſituate in the plane of the orbit *PAB*, will have the ſame effect as before; neither will it draw the body *P* from the plane of its orbit. But the other force *NM* acting in the direction of a line parallel to *ST* (and which therefore when the body *S* is without the line of the nodes is inclined to the plane of the orbit *PAB*) beſides the perturbation of the motion juſt now ſpoken of as to longitude, introduces another perturbation alſo as to latitude, attracting the body *P* out of the plane of its orbit. And this perturbation, in any given ſituation of the bodies *P* and *T* to each other, will be as the generating force *MN*; and therefore becomes leaſt when the force *MN* is leaſt, that is, (as was juſt now ſhewn) where the attraction *SN* is not much greater nor much leſs than the attraction *SK*. *Q. E. D.*

COR. 1. Hence it may be eaſily collected, that if ſeveral leſs bodies *P*, *S*, *R*, *&c.* revolve about a very great body *T*; the motion of the innermoſt revolving body *P* will be leaſt diſturbed by the attractions of the others, when the great body is as well attracted and agitated by the reſt (according to the ratio of the accelerative forces) as the reſt are by each other mutually.

COR. 2. In a ſyſtem of three bodies *T*, *P*, *S*, if the accelerative attractions of any two of them towards a third be to each other reciprocally as the ſquares of the diſtances; the body *P*, by the radius *PT*, will deſcribe its area ſwifter near the conjunction *A* and the oppoſition *B*, than it will near the quadratures *C* and *D*. For every force with which the body *P* is acted on and the body *T* is not, and which does not act in the direction of the line *PT*, does either accelerate or retard the deſcription of the area, according as it is directed, whether in conſequentia or in antecedentia. Such is the force *NM*. This force in the paſſage of the body *P* from *C* to *A* is directed in conſequentia to its motion, and therefore accelerates it; then as far as *D* in antecedentia, and retards the motion; then in conſequentia as far as *B*; and laſtly in antecedentia as it moves from *B* to *C*.

COR. 3. And from the ſame reaſoning it appears that the body *P*, *cæteris paribus*, moves more ſwiftly in the conjunction and oppoſition than in the quadratures.

COR. 4. The orbit of the body *P*, *cæteris paribus*, is more curve at the quadratures than at the conjunction and oppoſition. For the ſwifter bodies move, the leſs they deflect from a rectilinear path. And beſides the force *KL*, or *NM*, at the conjunction and oppoſition, is contrary to the force with which the body *T* attracts the body *P*; and therefore diminiſhes that force; but the body *P* will deflect the leſs from a rectilinear path the leſs it is impelled towards the body *T*.

COR. 5. Hence the body *P cæteris paribus* goes farther from the body *T* at the quadratures than at the conjunction and oppoſition. This is ſaid however, ſuppoſing no regard had to the motion of eccentricity. For if the orbit of the body *P* be eccentrical, its eccentricity (as will be ſhewn preſently by cor. 9.) will be greateſt when the apſides are in the ſyzygies; and thence it may ſometimes come to paſs, that the body *P* in its near approach to the farther apſis, may go farther from the body *T* at the ſyzygies, than at the quadratures.

COR. 6. Becauſe the centripetal force of the central body *T*, by which the body *P* is retained in its orbit, is increaſed at the quadratures by the addition cauſed by the force *LM*, and diminiſhed at the ſyzygies by the ſubduction cauſed by the force *KL*; and by reaſon the force *KL* is greater than *LM* is more diminiſhed than increaſed; and moreover ſince that centripetal force (by cor. 2. prop. 4.) is in a ratio compounded of the ſimple ratio of the radius *TP* directly, and the duplicate ratio of the periodical time inverſely; it is plain that this compounded ratio is diminiſhed by the action of the force *KL*; and therefore that the periodical time, ſuppoſing the radius of the orbit *PT* to remain the ſame, will be increaſed, and that in the ſubduplicate of that ratio in which the centripetal force is diminiſhed; and therefore ſuppoſing this radius increaſed or diminiſhed, the periodical time will be increaſed more or diminiſhed leſs than in the ſeſquiplicate ratio of this radius, by cor. 6. prop. 4. If that force of the central body ſhould gradually decay, the body *P* being leſs and leſs attracted would go farther and farther from the centre *T*; and on the contrary if it were increaſed it would draw nearer to it. Therefore if the action of the diſtant body *S*, by which that force is diminiſhed, were to increaſe and decreaſe by turns; the radius *TP* will be alſo increaſed and diminiſhed by turns; and the periodical time will be increaſed and diminiſhed in a ratio compounded of the ſeſquiplicate ratio of the radius, and of the ſubduplicate of that ratio in which the centripetal force of the central body *T* is diminiſhed or increaſed, by the increaſe or decreaſe of the action of the diſtant body *S*.

Cor. 7. It alſo follows from what was before laid down, that the axis of the ellipſis deſcribed by the body *P*, or the line of the apſides, does as to its angular motion go forwards and backwards by turns, but more forwards than backwards, and by the exceſs of its direct motion, is in the whole carried ſorwards. For the ſorce with which the body *P* is urged to the body *T* at the quadratures, where the force *MN* vaniſhes, is compounded of the ſorce *LM* and the centripetal force with which the body *T* attracts the body *P*. The firſt force *LM*, if the diſtance *PT* be increaſed, is increaſed in nearly the ſame proportion with that diſtance, and the other force decreaſes in the duplicate ratio of that diſtance; and therefore the ſum of theſe two forces decreaſes in a leſs than the duplicate ratio of the diſtance *PT*, and therefore by cor. 1. prop. 45. will make the line of the apſides, or, which is the ſame thing, the upper apſis, to go backward. But at the conjunction and oppoſition the force with which the body *P* is urged towards the body *T*, is the difference of the force *KL*, and of the force with which the body *T* attracts the body *P*; and that difference, becauſe the force *KL* is very nearly increaſed in the ratio of the diſtance *PT*, decreaſes in more than the duplicate ratio of the diſtance *PT*; and therefore by cor. 1. prop. 45. cauſes the line of the apſides to go forwards. In the places between the ſyzygies and the quadratures, the motion of the line of the apſides depends upon both theſe cauſes conjunctly, ſo that it either goes forwards or backwards in proportion to the exceſs of one of theſe cauſes above the other. Therefore ſince the force *KL* in the ſyzygies is almoſt twice as great as the force *LM* in the quadratures, the exceſs will be on the ſide of the force *KL*, and by conſequence the line of the apſides will be carried forwards. The truth of this and the foregoing corollary will be more eaſily underſtood by conceiving the ſyſtem of the two bodies *T* and *P*, to be ſurrounded on every ſide by ſeveral bodies *S*, *S*, *S*, *&c.* diſpoſed about the orbit *ESE*. For by the actions of theſe bodies the action of the body *T* will be diminiſhed on every ſide, and decreaſe in more than a duplicate ratio of the diſtance.

COR. 8. But ſince the progreſs or regreſs of the apſides depends upon the decreaſe of the centripetal force, that is, upon its being in a greater or leſs ratio than the duplicate ratio of the diſtance *TP*, in the paſſage of the body from the lower apſis to the upper; and upon a like increaſe in its return to the lower apſis again; and therefore becomes greateſt where the proportion of the force at the upper apſis to the force at the lower apſis recedes fartheſt from the duplicate ratio of the diſtances inverſely; it is plain that when the apſides are in the ſyzygies, they will, by reaſon of the ſubducting force *KL* or *NM*—*LM*, go forward more ſwiftly; and in the quadratures by the additional force *LM* go backward more ſlowly. When the velocity of the progreſs or ſlowneſs of the regreſs is continued for a long time, this inequality becomes exceeding great.

COR. 9. If a body is obliged, by a force reciprocally proportional to the ſquare of its diſtance from any centre, to revolve in an ellipſis round that centre; and afterwards in its deſcent from the upper apſis to the lower apſis, that force by a perpetual acceſſion of new force is increaſed in more than a duplicate ratio of the diminiſhed diſtance; it is manifeſt that the body being impelled always towards the centre by the perpetual acceſſion of this new force, will incline more towards that centre than if it were urged by that force alone which decreaſes in a duplicate ratio of the diminiſhed diſtance; and therefore will deſcribe an orbit interior to that elliptical orbit, and at the lower apſis approaching nearer to the centre than before. Therefore the orbit by the acceſſion of this new force will become more eccentrical. If now, while the body is returning from the lower to the upper apſis, it ſhould decreaſe by the ſame degrees by which it increaſed before, the body would return to its firſt diſtance; and therefore if the force decreaſes in a yet greater ratio, the body, being now leſs attracted than before, will aſcend to a ſtill greater diſtance, and ſo the eccentricity of the orbit will be increaſed ſtill more. Therefore if the ratio of the increaſe and decreaſe of the centripetal force be augmented each revolution, the eccentricity will be augmented alſo; and on the contrary, if that ratio decreaſe it will be diminiſhed.

Now therefore in the ſyſtem of the bodies *T*, *P*, *S*, when the apſides of the orbit *PAB* are in the quadratures, the ratio of that increaſe and decreaſe is leaſt of all, and becomes greateſt when the apſides are in the ſyzygies. If the apſides are placed in the quadratures, the ratio near the apſides is leſs, and near the ſyzygies greater, than the duplicate ratio of the diſtances, and from that greater ratio ariſes a direct motion of the line of the apſides, as was juſt now ſaid. But if we conſider the ratio of the whole increaſe or decreaſe in the progreſs between the apſides, this is leſs than the duplicate ratio of the diſtances. The force in the lower is to the force in the upper apſis, in leſs than a duplicate ratio of the diſtance of the upper apſis from the focus of the ellipſis to the diſtance of the lower apſis from the ſame focus; and contrarywiſe, when the apſides are placed in the ſyzygies the force in the lower apſis is to the force in the upper apſis in a greater than a duplicate ratio of the diſtances. For the forces *LM* in the quadratures added to the forces of the body *T* compoſe forces in a leſs ratio, and the forces *KL* in the ſyzygies ſubducted from the forces of the body *T* leave the forces in a greater ratio. Therefore the ratio of the whole increaſe and decreaſe in the paſſage between the apſides, is leaſt at the quadratures and greateſt at the ſyzygies; and therefore in the paſſage of the apſides from the quadratures to the ſyzygies it is continually augmented, and increaſes the eccentricity of the ellipſis; and in the paſſage from the ſyzygies to the quadratures it is perpetually decreaſing, and diminiſhes the eccentricity.

COR. 10. That we may give an account of the errors as to latitude, let us ſuppoſe the plane of the orbit *EST* to remain immoveable; and from the cauſe of the errors above explained it is manifeſt, that of the two forces *NM*, *ML* which are the only and entire cauſe of them, the force *ML* acting always in the plane of the orbit *PAB* never diſturbs the motions as to latitude; and that the force *NM*, when the nodes are in the ſyzygies, acting alſo in the ſame plane of the orbit, does not at that time affect thoſe motions. But when the nodes are in the quadratures, it diſturbs them very much, and attracting the body *P* perpetually out of the plane of its orbit, it diminiſhes the inclination of the plane in the paſſage of the body from the quadratures to the ſyzygies, and again increaſes the ſame in the paſſage from the ſyzygies to the quadratures. Hence it comes to paſs that when the body is in the ſyzygies the inclination is

then

then leaſt of all, and returns to the firſt magnitude nearly, when the body arrives at the next node. But if the nodes are ſituate at the octants after the quadratures, that is between *C* and *A*, *D* and *B*, it will appear from what was juſt now ſhewn that in the paſſage of the body *P* from either node to the ninetieth degree from thence, the inclination of the plane is perpetually diminiſhed; then in the paſſage through the next 45 degrees, to the next quadrature, the inclination is increaſed; and afterwards again, in its paſſage through another 45 degrees to the next node, it is diminiſhed. Therefore the inclination is more diminiſhed than increaſed, and is therefore always leſs in the ſubſequent node than in the preceding one. And by a like reaſoning, the inclination is more increaſed than diminiſhed, when the nodes are in the other octants between *A* and *D*, *B* and *C*. The inclination therefore is the greateſt of all when the nodes are in the ſyzygies. In their paſſage from the ſyzygies to the quadratures the inclination is diminiſhed at each appulſe of the body to the nodes; and becomes leaſt of all when the nodes are in the quadratures, and the body in the ſyzygies; then it increaſes by the ſame degrees by which it decreaſed before; and when the nodes come to the next ſyzygies returns to its former magnitude.

Cor. 11. Becauſe when the nodes are in the quadratures the body *P* is perpetually attracted from the plane of its orbit; and becauſe this attraction is made towards *S* in its paſſage from the node *C* through the conjunction *A* to the node *D*; and to the contrary part in its paſſage from the node *D* through the oppoſition *B* to the node *C*;

it is manifeſt that in its motion from the node *C*, the body recedes continually from the former plane *CD* of its orbit till it comes to the next node; and therefore at that node, being now at its greateſt diſtance from the firſt plane *CD*, it will paſs through the plane of the orbit *EST* not in *D*, the other node of that plane, but in a point that lies nearer to the body *S*, which therefore becomes a new place of the node in antecedentia to its former place. And by a like reaſoning, the nodes will continue to recede in their paſſage from this node to the next. The nodes therefore when ſituate in the quadratures recede perpetually, and at the ſyzygies, where no perturbation can be produced in the motion as to latitude, are quieſcent; in the intermediate places they partake of both conditions, and recede more ſlowly; and therefore being always either retrograde or ſtationary, they will be carried backwards, or in antecedentia, each revolution.

COR. 12. All the errors deſcribed in theſe corollaries are a little greater at the conjunction of the bodies *P*, *S*, than at their oppoſition; becauſe the generating forces *NM* and *ML* are greater.

COR. 13. And ſince the cauſes and proportions of the errors and variations mentioned in theſe corollaries do not depend upon the magnitude of the body *S*, it follows that all things before demonſtrated will happen, if the magnitude of the body *S* be imagined ſo great as that the ſyſtem of the two bodies *P* and *T* may revolve about it. And from this increaſe of the body *S*, and the conſequent increaſe of its centripetal force from which the errors of the body *P* ariſe, it will follow that all theſe errors, at equal diſtances, will be greater in this caſe, than in the other where the body *S* revolves about the ſyſtem of the bodies *P* and *T*.

Cor. 14. But ſince the forces *NM*, *ML*, when the body *S* is exceedingly diſtant, are very nearly as the force *SK* and the ratio of *PT* to *ST* conjunctly; that is, if both the diſtance *PT*, and the abſolute force of the body *S* be given, as ST^3 reciprocally; and ſince thoſe forces *NM*, *ML* are the cauſes of all the errors and effects treated of in the foregoing corollaries; it is manifeſt, that all thoſe effects, if the ſyſtem of bodies *T* and *P* continue as before, and only the diſtance *ST* and the abſolute force of the body *S* be changed, will be very nearly in a ratio compounded of the direct ratio of the abſolute force of the body *S*, and the triplicate inverſe ratio of the diſtance *ST*. Hence if the ſyſtem of bodies *T* and *P* revolve about a diſtant body *S*; thoſe forces *NM*, *ML* and their effects will be (by cor. 2 and 6. prop. 4.) reciprocally in a duplicate ratio of the periodical time. And thence alſo if the magnitude of the body *S* be proportional to its abſolute force, thoſe forces *NM*, *ML*, and their effects, will be directly as the cube of the apparent diameter of the diſtant body *S* viewed from *T*, and ſo *vice versâ*. For theſe ratio's are the ſame as the compounded ratio above-mentioned.

Cor. 15. And becauſe if the orbits *ESE* and *PAB*, retaining their figure, proportions and inclination to each other, ſhould alter their magnitude; and the forces of the bodies *S* and *T* ſhould either remain, or be changed in any given ratio; theſe forces (that is, the force of the body *T* which obliges the body *P* to deflect from a rectilinear

linear courſe into the orbit *PAB*, and the force of the body *S*, which cauſes the body *P* to deviate from that orbit) would act always in the ſame manner, and in the ſame proportion; it follows that all the effects will be ſimilar and proportional, and the times of thoſe effects proportional alſo; that is, that all the linear errors will be as the diameters of the orbits, the angular errors the ſame as before; and the times of ſimilar linear errors, or equal angular errors as the periodical times of the orbits.

COR. 16. Therefore if the figures of the orbits and their inclination to each other be given, and the magnitudes, forces, and diſtances of the bodies be any how changed; we may, from the errors and times of thoſe errors in one caſe, collect very nearly the errors and times of the errors in any other caſe. But this may be done more expeditiouſly by the following method. The forces *NM*, *ML*, other things remaining unaltered, are as the radius *TP*; and their periodical effects (by cor. 2. lem. 10.) are as the forces, and the ſquare of the periodical time of the body *P* conjunctly. Theſe are the linear errors of the body *P*; and hence the angular errors as they appear from the centre *T* (that is the motion of the apſides and of the nodes, and all the apparent errors as to longitude and latitude) are in each revolution of the body *P*, as the ſquare of the time of the revolution very nearly. Let theſe ratio's be compounded with the ratio's in cor. 14. and in any ſyſtem of bodies *T*, *P*, *S*, where *P* revolves about *T* very near to it, and *T* revolves about *S* at a great diſtance, the angular errors of the body *P*, obſerved from the centre *T*, will be in each revolution of the body *P* as the ſquare of the periodical time of the body *P* directly, and the ſquare of the periodical time of the body *T* inverſely. And therefore the mean motion of the line of the apſides will be in a given ratio to the mean motion of the nodes; and both thoſe motions will be as the periodical time of the body *P* directly, and the ſquare of the periodical time of the body *T* inverſely. The increaſe or diminution of the eccentricity and inclination of the orbit *PAB* makes no ſenſible variation in the motions of the apſides and nodes, unleſs that increaſe or diminution be very great indeed.

Cor. 17. Since the line *LM* becomes ſometimes greater and ſometimes leſs than the radius *PT*, let the mean quantity of the force *LM* be expreſſed by that radius *PT*; and then that mean force will be to the mean force *SK* or *SN* (which may be alſo expreſſed by *ST*) as the length *PT* to the length *ST*. But the mean force *SN* or *ST*, by which the body *T* is retained in the orbit it deſcribes about *S*, is to the force with which the body *P* is retained in its orbit about *T*, in a ratio compounded of the ratio of the radius *ST* to the radius *PT* and the duplicate ratio of the periodical time of the body *P* about *T* to the periodical time of the body *T* about *S*. And ex æquo, the mean force *LM* is to the force by which the body *P* is retained in its orbit about *T* (or by which the ſame body *P* might revolve at the diſtance *PT* in the ſame periodical time about any immoveable point *T*) in the ſame duplicate ratio of the periodical times. The periodical times therefore being given, together with the diſtance *PT*, the mean force *LM* is alſo given; and that force being given, there is given alſo the force *MN* very nearly, by the analogy of the lines *PS*, and *MN*.

Cor. 18. By the ſame laws by which the body *P* revolves about the body *T*, let us ſuppoſe many fluid bodies to move round *T* at equal diſtances from it; and to be ſo numerous that they may all become contiguous to each other, ſo as to form a fluid annulus or ring, of a round figure and concentrical to the body *T*; and the ſeveral parts of this annulus, performing their motions by the ſame law as the body *P*, will draw nearer to the body *T* and move ſwifter in the conjunction and oppoſition of themſelves and the body *S*, than in the quadratures. And the nodes of this annulus, or its interſections with the plane of the orbit of the body *S*, or *T*, will reſt at the ſyzygies; but out of the ſyzygies they will be carried backward, or in antecedentia; with the greateſt ſwiftneſs in the quadratures, and more ſlowly in other places. The inclination of this annulus alſo will vary, and its axis will oſcillate each revolution, and when the revolution is compleated will return to its former ſituation, except only that it will be carried round a little by the præceſſion of the nodes.

Cor. 19. Suppoſe now the ſphærical body *T*, conſiſting of ſome matter not fluid, to be enlarged, and to extend it ſelf on every ſide as far as that annulus, and that a channel were cut all round its circumference containing water; and that this ſphere revolves uniformly about its own axis in the ſame periodical time. This water being accelerated and retarded by turns (as in the laſt corollary) will be ſwifter at the ſyzygies, and ſlower at the quadratures than the ſurface of the globe, and ſo will ebb and flow in its channel after the manner of the Sea. If the attraction of the body *S* were taken away; the water would acquire no motion of flux and reflux by revolving round the quieſcent centre of the globe. The caſe is the ſame of a globe moving uniformly forwards in a right line, and in the mean time revolving about its centre, (by cor. 5. of the laws of motion) and of a globe uniformly attracted from its rectilinear courſe (by cor. 6. of the ſame laws.) But let the body *S* come to act upon it, and by its unequable attraction the water will receive this new motion. For there will be a ſtronger attraction upon that part of the water that is neareſt to the body, and a weaker upon that part which is more remote. And the force *LM* will attract the water downwards at the quadratures, and depreſs it as far as the ſyzygies; and the force *KL* will attract it upwards in the ſyzygies, and withhold its deſcent, and make it riſe as far as the quadratures; except only in ſo far as the motion of flux and reflux may be directed by the channel of the water, and be a little retarded by friction.

Cor. 20. If now the annulus becomes hard, and the globe is diminiſhed, the motion of flux and reflux will ceaſe; but the oſcillating motion of the inclination and the præceſſion of the nodes will remain. Let the globe have the ſame axis with the annulus and perform its revolutions in the ſame times, and at its ſurface touch the annulus within, and adhere to it; then, the globe partaking of the motion of the annulus, this whole compages will oſcillate, and the nodes will go backward. For the globe, as we ſhall ſhew preſently, is perfectly indifferent to the receiving of all impreſſions. The greateſt angle of the inclination of the annulus ſingle, is when the nodes are in the ſyzygies. Thence in the progreſs of the nodes to the quadratures, it endeavours to diminiſh its inclination and by that endeavour impreſſes a motion upon the whole globe. The globe retains this motion impreſſed, till the annulus by a contrary endeavour deſtroys that motion and impreſſes a new motion in a contrary direction. And by this means the greateſt motion of the decreaſing inclination happens when the nodes are in the quadratures; and the leaſt angle of inclination in the octants after the quadratures; and again, the greateſt motion of reclination happens when the nodes are in the ſyzygies; and the greateſt angle of reclination in the octants following. And the caſe is the ſame of a globe without this annulus, if it be a little higher or a little denſer in the æquatorial than in the polar regions. For the exceſs of that matter in the regions near the æquator ſupplies the place of the annulus. And though we ſhould ſuppoſe the centripetal force of this globe to be any how increaſed ſo that all its parts were to tend downwards, as the parts of our Earth gravitate to the centre, yet the phænomena of this and the preceding corollary would ſcarce be altered; except that the places of the greateſt and leaſt height of the water will be different. For the water is now no longer ſuſtained and kept in its orbit by its centrifugal force, but by the channel in which it flows. And beſides the force LM attracts the water downwards moſt in the quadratures, and the force KL or $NM-LM$ attracts it upwards moſt in the ſyzygies. And theſe forces conjoined ceaſe to attract the water downwards, and begin to attract it upwards in the octants before the ſyzygies; and ceaſe to attract the water upwards, and begin to attract the water downwards in the octants after the ſyzygies. And thence the greateſt height of the water may happen about the octants after the ſyzygies; and the leaſt height about the octants after the quadratures; excepting only ſo far as the motion of aſcent or deſcent impreſſed by theſe forces may by the vis inſita of the water continue a little longer, or be ſtopt a little ſooner by impediments in its channel.

Cor. 21. For the ſame reaſon that redundant matter in the æquatorial regions of a globe cauſes the nodes to go backwards, and therefore by the increaſe of that matter that retrogradation is increaſed, by the diminution is diminiſhed, and by the removal quite ceaſes; it follows, that if more than that redundant matter be taken away, that is, if the globe be either more depreſſed, or of a more rare conſiſtence near the æquator than near the poles, there will ariſe a motion of the nodes in conſequentia.

Cor. 22. And thence from the motion of the nodes is known the conſtitution of the globe. That is if the globe retains unalterably the ſame poles, and the motion (of the nodes) be in antecedentia, there is a redundance of the matter near the equator; but if in conſequentia, a deficiency. Suppoſe an uniform and exactly ſphærical globe to be firſt at reſt in a free ſpace; then by ſome impulſe made obliquely upon its ſuperficies to be driven from its place, and to receive a motion, partly circular and partly right forward. Becauſe this globe is perfectly indifferent to all the axes that paſs through its centre, nor has a greater propenſity to one axis or to one ſituation of the axis than to any other, it is manifeſt that by its own force it will never change its axis, or the inclination of it. Let now this globe be impelled obliquely by a new impulſe in the ſame part of its ſuperficies as before; and ſince the effect of an impulſe is not at all changed by its coming ſooner or later, it is manifeſt that theſe two impulſes ſucceſſively impreſſed will produce the ſame motion, as if they were impreſſed at the ſame time; that is, the ſame motion as if the globe had been impelled by a ſimple force compounded of them both (by cor. 2. of the laws) that is a ſimple motion about an axis of a given inclination. And the caſe is the ſame if the ſecond impulſe were made upon any other place of the æquator of the firſt motion; and alſo if the firſt impulſe were made upon any place in the æquator of the motion which would be generated by the ſecond impulſe alone; and therefore alſo when both impulſes are made in any places whatſoever; for theſe impulſes will generate the ſame circular motion, as if they were impreſſed together and at once in the place of the interſections of the æquators of thoſe motions, which would be generated by each of them ſeparately. Therefore a homogeneous and perfect globe will not retain ſeveral diſtinct motions, but will unite all thoſe that are impreſſed on it, and reduce them into one; revolving, as far as in it lies, always with a ſimple and uniform motion about one ſingle given axis, with an inclination perpetually invariable. And the inclination of the axis, or the velocity of the rotation will not be changed by centripetal force. For if the globe be ſuppoſed to be divided into two hemiſpheres, by any plane whatſoever paſſing through its own centre and the centre to which the force is directed; that force will always urge each hemiſphere equally; and therefore will not incline the globe any way as to its motion round its own axis. But let there be added any where between the pole and the æquator a heap of new matter like a mountain, and this by its perpetual endeavour to recede from the centre of its motion, will diſturb the motion of the globe, and cauſe its poles to wander about its ſuperficies, deſcribing circles about themſelves and their oppoſite points. Neither can this enormous evagation of the poles be corrected, unleſs by placing that mountain either in one of the poles, in which caſe by cor. 21. the nodes of the æquator will go forwards; or in the æquatorial regions, in which caſe by cor. 20. the nodes will go backward; or laſtly by adding on the other ſide of the axis a new quantity of matter, by which the mountain may be ballanced in its motion; and then the nodes will either go forwards or backwards, as the mountain and this newly added matter happen to be nearer to the pole or to the equator.

PROPOSITION LXVII. THEOREM XXVII.

The ſame laws of attraction being ſuppoſed, I ſay that the exterior body S *does, by radij drawn to the point* O, *the common centre of gravity of the interior bodies* P *and* T, *deſcribe round that centre areas more proportional*

tional to the times, and an orbit more approaching to the form of an ellipſis having its focus in that centre, than it can deſcribe round the innermoſt and greateſt body T *by radij drawn to that body.*

For the attractions of the body *S* (*Pl.* 21. *Fig.* 3.) towards *T* and *P* compoſe its abſolute attraction, which is more directed towards *O* the common centre of gravity of the bodies *T* and *P*, than it is to the greateſt body *T*; and which is more in a reciprocal proportion to the ſquare of the diſtance *SO*, than it is to the ſquare of the diſtance *ST*; as will eaſily appear by a little conſideration.

PROPOSITION LXVIII. THEOREM XXVIII.

The ſame laws of attraction ſuppoſed, I ſay that the exterior body S *will, by radij drawn to* O *the common centre of gravity of the interior bodies* P *and* T, *deſcribe round that centre, area's more proportional to the times, and an orbit more approaching to the form of an ellipſis having its focus in that centre, if the innermoſt and greateſt body be agitated by theſe attractions as well as the reſt, than it would do if that body were either* *at reſt as not attracted, or were much more or much leſs attracted or much more or much leſs agitated.*

This may be demonſtrated after the ſame manner as prop. 66. but by a more prolix reaſoning, which I therefore paſs over. It will be ſufficient to conſider it after this manner. From the demonſtration of the laſt propoſition it is plain, that the centre, towards which the body *S* is urged by the two forces conjunctly, is very near to the common centre of gravity of thoſe two other bodies. If this centre were to coincide with that common centre, and moreover the common centre of gravity of all the three bodies were at reſt; the body *S* on one ſide, and the common centre of gravity of the other two bodies on the other ſide, would deſcribe true ellipſes about that quieſcent common centre. This appears from cor. 2. prop. 58. compared with what was demonſtrated in prop. 64 and 65. Now this accurate elliptical motion will be diſturbed a little by the diſtance of the centre of the two bodies from the centre towards which the third body *S* is attracted. Let there be added moreover a motion to the common centre of the three, and the perturbation will be increaſed yet more. Therefore the perturbation is leaſt when the common centre of the three bodies is at reſt; that is, when the innermoſt and greateſt body *T* is attracted according to the ſame law as the reſt are; and is always greateſt, when the common centre of the three, by the diminution of the motion of the body *T*, begins to be moved, and is more and more agitated.

COR. And hence if more leſſer bodies revolve about the great one, it may eaſily be inferred that the orbits deſcribed will approach nearer to ellipſes, and the deſcriptions of area's will be more nearly equable, if all the bodies mutually attract and agitate each other with accelerative forces that are as their abſolute forces directly, and the ſquares of the diſtances inverſely; and if the focus of each orbit be placed in the common centre of gravity of all the interior bodies; (that is, if the focus of the firſt and innermoſt orbit be placed in the centre of gravity of the greateſt and innermoſt body; the focus of the ſecond orbit in the common centre of gravity of the two innermoſt bodies; the focus of the third orbit in the common centre of gravity of the three innermoſt; and ſo on) than if the innermoſt body were at reſt, and was made the common focus of all the orbits.

PROPOSITION LXIX. THEOREM XXIX.

In a ſyſtem of ſeveral bodies A, B, C, D, *&c. if any one of thoſe bodies as* A, *attract all the reſt*, B, C, D, *&c. with accelerative forces that are reciprocally as the ſquares of the diſtances from the attracting body; and another body as* B *attracts alſo the reſt*, A, C, D, *&c. with forces that are reciprocally as the ſquares of the diſtances from the attracting body; the* *abſolute forces of the attracting bodies* A *and* B *will be to each other, as thoſe very bodies* A *and* B *to which thoſe forces belong.*

For the accelerative attractions of all the bodies *B, C, D,* towards *A* are by the ſuppoſition equal to each other at equal diſtances; and in like manner the accelerative attractions of all the bodies towards *B* are alſo equal to each other at equal diſtances. But the abſolute attractive force of the body *A* is to the abſolute attractive force of the body *B*, as the accelerative attraction of all the bodies towards *A* to the accelerative attraction of all the bodies towards *B* at equal diſtances; and ſo is alſo the accelerative attraction of the body *B* towards *A*, to the accelerative attraction of the body *A* towards *B*. But the accelerative attraction of the body *B* towards *A* is to the accelerative attraction of the body *A* towards *B* as the maſs of the body *A* to the maſs of the body *B*; becauſe the motive forces which (by the 2d, 7th, and 8th definition) are as the accelerative forces and the bodies attracted conjunctly, are here equal to one another by the third law. Therefore the abſolute attractive force of the body *A* is to the abſolute attractive force of the body *B* as the maſs of the body *A* to the maſs of the body *B*. *Q. E. D.*

Cor. 1. Therefore if each of the bodies of the ſyſtem *A, B, C, D, &c.* does ſingly attract all the reſt with accelerative forces that are reciprocally as the ſquares of the diſtances from the attracting body; the abſolute forces of all thoſe bodies will be to each other as the bodies themſelves.

COR. 2. By a like reaſoning if each of the bodies of the ſyſtem *A*, *B*, *C*, *D*, *&c.* do ſingly attract all the reſt with accelerative forces, which are either reciprocally or directly in the ratio of any power whatever of the diſtances from the attracting body; or which are defined by the diſtances from each of the attracting bodies according to any common law; it is plain that the abſolute forces of thoſe bodies are as the bodies themſelves.

COR. 3. In a ſyſtem of bodies whoſe forces decreaſe in the duplicate ratio of the diſtances, if the leſſer revolve about one very great one in ellipſes, having their common focus in the centre of that great body, and of a figure exceeding accurate; and moreover by radij drawn to that great body deſcribe area's proportional to the times exactly; the abſolute forces of thoſe bodies to each other will be either accurately or very nearly in the ratio of the bodies. And ſo on the contrary. This appears from cor. of prop. 68. compared with the firſt corollary of this prop.

SCHOLIUM.

Theſe propoſitions naturally lead us to the analogy there is between centripetal forces, and the central bodies to which thoſe forces uſe to be directed. For it is reaſonable to ſuppoſe that forces which are directed to bodies ſhould depend upon the nature and quantity of thoſe bodies, as we ſee they do in magnetical experiments. And when ſuch caſes occur, we are to compute the attractions of the bodies by aſſigning to each of their particles its proper force, and then collecting the ſum of them all. I here uſe the word attraction in general for any endeavour, of what kind ſoever, made by bodies to approach to each other; whether that endeavour ariſe from the action of the bodies themſelves as tending mutually to, or agitating each other by ſpirits emitted; or whether it ariſes from the action of the æther or of the air, or of any medium whatſoever, whether corporeal or incorporeal, any how impelling bodies placed therein towards each other. In the ſame general ſenſe I uſe the word impulſe, not defining in this treatiſe the ſpecies or phyſical qualities of forces, but inveſtigating the quantities and mathematical proportions of them; as I obſerved before in the definitions. In mathematics we are to inveſtigate the quantities of forces with their proportions conſequent upon any conditions ſuppoſed; then when we enter upon phyſics, we compare thoſe proportions with the phænomena of Nature; that we may know what conditions of thoſe forces anſwer to the ſeveral kinds of attractive bodies. And this preparation being made, we argue more ſafely concerning the phyſical ſpecies, cauſes, and proportions of the forces. Let us ſee then with what forces ſphærical bodies conſiſting of particles endued with attractive powers in the manner above ſpoken of muſt act mutually upon one another; and what kind of motions will follow from thence.

SECTION XII.

Of the attractive forces of ſphærical bodies.

Proposition LXX. Theorem XXX.

If to every point of a ſphærical ſurface there tend equal centripetal forces decreaſing in the duplicate ratio of the diſtances from thoſe points; I ſay that a corpuſcle placed within that ſuperficies will not be attracted by thoſe forces any way.

Let *HIKL* (*Pl.* 21. *Fig.* 4.) be that ſphærical ſuperficies, and *P* a corpuſcle placed within. Through *P* let there be drawn to this ſuperficies the two lines *HK*, *IL*, intercepting very ſmall arcs *HI*, *KL*; and becauſe (by cor. 3. lem. 7.) the triangles *HPI*, *LPK* are alike, thoſe arcs will be propor-

tional

tional to the diſtances *HP*, *LP*; and any particles at *HI* and *KL* of the ſphærical ſuperficies, terminated by right lines paſſing through *P*, will be in the duplicate ratio of thoſe diſtances. Therefore the forces of theſe particles exerted upon the body *P* are equal between themſelves. For the forces are as the particles directly and the ſquares of the diſtances inverſely. And theſe two ratio's compoſe the ratio of equality. The attractions therefore being made equally towards contrary parts deſtroy each other. And by a like reaſoning all the attractions through the whole ſphærical ſuperficies are deſtroyed by contrary attractions. Therefore the body *P* will not be any way impelled by thoſe attractions. *Q. E. D.*

PROPOSITION LXXI. THEOREM XXXI.

The ſame things ſuppoſed as above, I ſay that a corpuſcle placed without the ſphærical ſuperficies is attracted towards the centre of the ſphere with a force reciprocally proportional to the ſquare of its diſtance from that centre.

Let *AHKB*, *ahkb* (*Pl.* 21. *Fig.* 5.) be two equal ſphærical ſuperficies deſcribed about the centres *S*, *s*; their diameters *AB*, *ab*; and let *P* and *p* be two corpuſcles ſituate without the ſpheres in thoſe diameters produced. Let there be drawn from the corpuſcles the lines *PHK*, *PIL*, *phk*, *pil*, cutting off from the great circles AHB, ahb, the equal arcs HK, hk, IL, il; and to thoſe lines let fall the perpendiculars SD, sd, SE, se, IR, ir; of which let SD, sd cut PL, pl in F and f. Let fall alſo to the diameters the perpendiculars IQ, iq. Let now the angles DPE, dpe vaniſh; and becauſe DS and ds, ES and es are equal, the lines PE, PF, and pe, pf, and the lineolæ DF, df may be taken for equal; becauſe their laſt ratio, when the angles DPE, dpe vaniſh together, is the ratio of equality. Theſe things then ſuppoſed, it will be, as PI to PF ſo is RI to DF, and as pf to pi ſo is df or DF to ri; and *ex æquo*, as $PI \times pf$ to $PF \times pi$ ſo is RI to ri, that is (by cor. 3. lem. 7.) ſo is the arc IH to the arc ih. Again PI is to PS as IQ to SE, and ps to pi as se or SE to iq; and *ex æquo* $PI \times ps$ to $PS \times pi$ as IQ to iq. And compounding the ratio's $PI^2 \times pf \times ps$ is to $pi^2 \times PF \times PS$, as $IH \times IQ$ to $ih \times iq$; that is, as the circular ſuperficies which is deſcribed by the arc IH as the ſemicircle AKB revolves about the diameter AB, is to the circular ſuperficies deſcribed by the arch ih as the ſemicircle akb revolves about the diameter ab. And the forces with which theſe ſuperficies attract the corpuſcles P and p in the direction of lines tending to thoſe ſuperficies are by the hypotheſis as the ſuperficies themſelves directly, and the ſquares of the diſtances of the ſuperficies from thoſe corpuſcles inverſely; that is, as $pf \times ps$ to $PF \times PS$. And theſe forces again are to the oblique parts of them which (by the reſolution of forces as in cor. 2. of the laws) tend to the centres in the directions of the lines PS, ps, as PI to PQ, and pi to pq; that is (becauſe of the like triangles PIQ and PSF, piq and psf) as PS to PF and ps to pf. Thence *ex æquo*, the attraction of the corpuſcle P towards S is to the attraction of the corpuſcle p towards s, as $\frac{PF \times pf \times ps}{PS}$ is to $\frac{pf \times PF \times PS}{ps}$, that is, as ps^2 to PS^2. And by a like reaſoning the forces with which the ſuperficies deſcribed by the revolution of the arcs KL, kl attract thoſe corpuſcles, will be as ps^2 to PS^2. And in the ſame ratio will be the forces of all the circular ſuperficies into which each of the ſphærical ſuperficies may be divided by taking sd always equal to SD, and se equal to SE. And therefore by compoſition, the forces of the entire ſphærical ſuperficies exerted upon thoſe corpuſcles will be in the ſame ratio. *Q. E. D.*

PROPOSITION LXXII. THEOREM XXXII.

If to the ſeveral points of a ſphere there tend equal centripetal forces decreaſing in a duplicate ratio of the diſtances from thoſe points; and there be given both the denſity of the ſphere and the ratio of the diameter of the ſphere to the diſtance of the corpuſcle from its centre; I ſay that the force with which the corpuſcle is attra-

cted is proportional to the ſemi-diameter of the ſphere.

For conceive two corpuſcles to be ſeverally attracted by two ſpheres, one by one the other by the other, and their diſtances from the centres of the ſpheres to be proportional to the diameters of the ſpheres reſpectively; and the ſpheres to be reſolved into like particles diſpoſed in a like ſituation to the corpuſcles. Then the attractions of one corpuſcle towards the ſeveral particles of one ſphere, will be to the attractions of the other towards as many analogous particles of the other ſphere in a ratio compounded of the ratio of the particles directly and the duplicate ratio of the diſtances inverſely. But the particles are as the ſpheres, that is in a triplicate ratio of the diameters, and the diſtances are as the diameters; and the firſt ratio directly with the laſt ratio taken twice inverſely, becomes the ratio of diameter to diameter. *Q. E. D.*

COR. 1. Hence if corpuſcles revolve in circles about ſpheres compoſed of matter equally attracting; and the diſtances from the centres of the ſpheres be proportional to their diameters; the periodic times will be equal.

COR. 2. And *vice versâ*, if the periodic times are equal, the diſtances will be proportional to the diameters. Theſe two corollaries appear from cor. 3. prop. 4.

COR. 3. If to the ſeveral points of any two ſolids whatever, of like figure and equal denſity, there tend equal centripetal forces decreaſing in a duplicate ratio of the diſtances from thoſe points; the forces with which corpuſcles placed in a like ſituation to thoſe two ſolids, will be attracted by them will be to each other as the diameters of the ſolids.

PROPOSITION LXXIII. THEOREM XXXIII.

If to the ſeveral points of a given ſphere there tend equal centripetal forces decreaſing in a duplicate ratio of the diſtances from the points; I ſay that a corpuſcle placed within the ſphere is attracted by a force proportional to its diſtance from the centre.

In the ſphere *ABCD* (*Pl.* 21. *Fig.* 6.) deſcribed about the centre *S*, let there be placed the corpuſcle *P*; and about the ſame centre *S*, with the interval *SP*, conceive deſcribed an interior ſphere *PEQF*. It is plain (by prop. 70.) that the concentric ſphærical ſuperficies of which the difference *AEBF* of the ſpheres is compoſed, have no effect at all upon the body *P*; their attractions being deſtroyed by contrary attractions. There remains therefore only the attraction of the interior ſphere *PEQF*. And (by prop. 72.) this is as the diſtance *PS*. *Q. E. D.*

SCHOLIUM.

By the ſuperficies of which I here imagine the ſolids compoſed, I do not mean ſuperficies purely mathe-

Plate XXI. *Vol.* I. *P.* 268.

Fig. 1. *p.* 228.

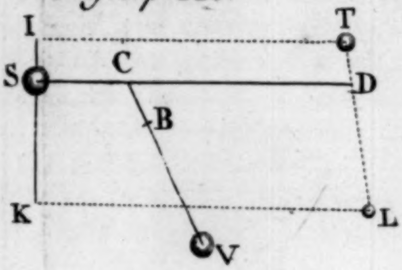


Fig. 2 *p.* 235.

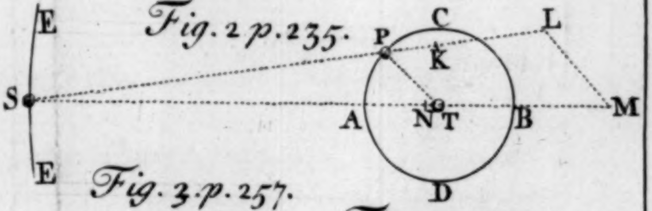


Fig. 3. *p.* 257.

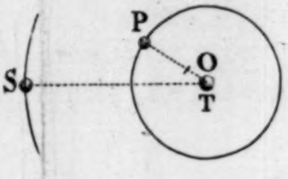


Fig. 4. *p.* 263.

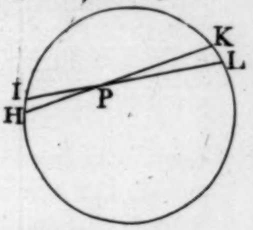


Fig. 5. *p.* 264.

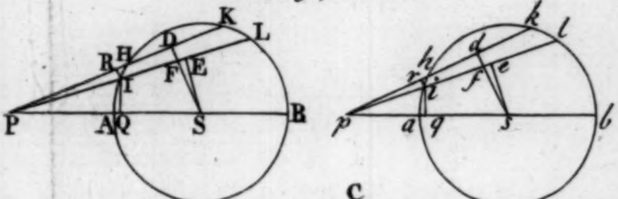
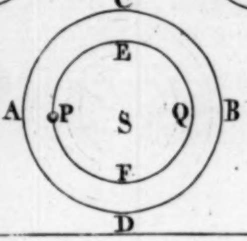


Fig. 6.
p. 268.



mathematical, but orbs ſo extreamly thin, that their thickneſs is as nothing; that is, the evaneſcent orbs, of which the ſphere will at laſt conſiſt, when the number of the orbs is increaſed, and their thickneſs diminiſhed without end. In like manner, by the points of which lines, ſurfaces and ſolids are ſaid to be compoſed, are to be underſtood equal particles whoſe magnitude is perfectly inconſiderable.

Proposition LXXIV. Theorem XXXIV.

The ſame things ſuppoſed, I ſay that a corpuſcle ſituate without the ſphere is attracted with a force reciprocally proportional to the ſquare of its diſtance from the centre.

For ſuppoſe the ſphere to be divided into innumerable concentric ſphærical ſuperficies, and the attractions of the corpuſcle ariſing from the ſeveral ſuperficies will be reciprocally proportional to the ſquare of the diſtance of the corpuſcle from the centre of the ſphere (by prop. 71.) And by compoſition, the ſum of thoſe attractions, that is, the attraction of the corpuſcle towards the entire ſphere, will be in the ſame ratio. *Q. E. D.*

Cor. 1. Hence the attractions of homegeneous ſpheres at equal diſtances from the centres will be as the ſpheres themſelves. For (by prop. 72) if the diſtances be proportional to the diameters of the ſpheres, the forces will be as the diameters. Let the greater diſtance be diminiſhed in that ratio; and the diſtances now being equal, the attraction will be increaſed in the duplicate of that ratio; and therefore will be to the other attraction in the triplicate of that ratio; that is, in the ratio of the ſpheres.

Cor. 2. At any diſtances whatever, the attractions are as the ſpheres applied to the ſquares of the diſtances.

Cor. 3. If a corpuſcle placed without an homogeneous ſphere is attracted by a force reciprocally proportional to the ſquare of its diſtance from the centre, and the ſphere conſiſts of attractive particles; the force of every particle will decreaſe in a duplicate ratio of the diſtance from each particle.

PROPOSITION LXXV. THEOREM XXXV.

If to the ſeveral points of a given ſphere there tend equal centripetal forces decreaſing in a duplicate ratio of the diſtances from the points; I ſay that another ſimilar ſphere will be attracted by it with a force reciprocally proportional to the ſquare of the diſtance of the centres.

For the attraction of every particle is reciprocally as the ſquare of its diſtance from the centre of the attracting ſphere (by prop. 74.) and is therefore the ſame as if that whole attracting force iſſued from one ſingle corpuſcle placed in the centre of this ſphere. But this attraction is as great, as on the other hand the attraction of the ſame corpuſcle would be, if that were it ſelf attracted by the ſeveral particles of the attracted ſphere with the ſame force with which they are attracted by it. But that attraction of the corpuſcle would be (by prop. 74.) reciprocally proportional to the ſquare of its diſtance from the centre of the ſphere; therefore the attraction of the ſphere, equal thereto, is alſo in the ſame ratio. *Q. E. D.*

COR. 1. The attractions of ſpheres towards other homogeneous ſpheres, are as the attracting ſpheres applied to the ſquares of the diſtances of their centres from the centres of thoſe which they attract.

COR. 2. The caſe is the ſame when the attracted ſphere does alſo attract. For the ſeveral points of the one attract the ſeveral points of the other with the ſame force with which they themſelves are attracted by the others again; and therefore ſince in all attractions (by law 3.) the attracted and attracting point are both equally acted on, the force will be doubled by their mutual attractions, the proportions remaining.

COR. 3. Thoſe ſeveral truths demonſtrated above concerning the motion of bodies about the focus of the conic ſections, will take place when an attracting ſphere is placed in the focus, and the bodies move without the ſphere.

COR. 4. Thoſe things which were demonſtrated before of the motion of bodies about the centre of the conic ſections take place when the motions are performed within the ſphere.

PROPOSITION LXXVI. THEOREM XXXVI.

If ſpheres be however diſſimilar (as to denſity of matter and attractive force) in the progreſs right onward from the centre to the circumference; but every where ſimilar, at every given diſtance from the centre, on all ſides round about; and the attractive force of every point decreaſes in the duplicate ratio of the diſtance of the body attracted; I ſay that the whole force with which one of theſe ſpheres attracts the other, will be reciprocally proportional to the ſquare of the diſtance of the centres.

Imagine ſeveral concentric ſimilar ſpheres, *AB*, *CD*, *EF*, *&c.* (*Pl.* 22. *Fig.* 1.) the innermoſt of which added to the outermoſt may compoſe a matter more denſe towards the centre, or ſubducted from them may leave the ſame more lax and rare. Then by prop. 75. theſe ſpheres will attract other ſimilar concentric ſpheres *GH*, *IK*, *LM*, *&c.* each the other, with forces reciprocally proportional to the ſquare of the diſtance *SP*. And by compoſition or diviſion, the ſum of all thoſe forces, or the exceſs of any of them above the others; that is, the entire force with which the whole ſphere *AB* (compoſed of any concentric ſpheres or of their differences) will attract the whole ſphere *G H* (compoſed of any concentric ſpheres or their differences) in the ſame ratio. Let the number of the concentric ſpheres be increaſed in infinitum, ſo that the denſity of the matter together with the attractive force may, in the progreſs from the circumference to the centre, increaſe or decreaſe according to any given law; and by the addition of matter not attractive let the deficient denſity be ſupplied that ſo the ſpheres may acquire any form deſired; and the force with which one of theſe attracts the other, will be ſtill, by the former reaſoning, in the ſame ratio of the ſquare of the diſtance inverſely. *Q. E. D.*

COR. 1. Hence if many ſpheres of this kind, ſimilar in all reſpects, attract each other mutually; the accelerative attractions of each to each, at any equal diſtances of the centres, will be as the attracting ſpheres.

COR. 2. And at any unequal diſtances, as the attracting ſpheres applied to the ſquares of the diſtances between the centres.

COR. 3. The motive attractions, or the weights of the ſpheres towards one another will be at equal diſtances of the centres as the attracting and attracted ſpheres conjunctly; that is, as the products ariſing from multiplying the ſpheres into each other.

COR. 4. And at unequal diſtances, as thoſe products directly and the ſquares of the diſtances between the centres inverſely.

COR. 5. Theſe proportions take place alſo, when the attraction ariſes from the attractive virtue of both ſpheres mutually exerted upon each other. For the attraction is only doubled by the conjunction of the forces, the proportions remaining as before.

Cor. 6. If ſpheres of this kind revolve about others at reſt, each about each; and the diſtances between the centres of the quieſcent and revolving bodies are proportional to the diameters of the quieſcent bodies; the periodic times will be equal.

Cor. 7. And again, if the periodic times are equal, the diſtances will be proportional to the diameters.

Cor. 8. All thoſe truths above demonſtrated, relating to the motions of bodies about the foci of conic ſections, will take place, when an attracting ſphere, of any form and condition like that above deſcribed, is placed in the focus.

Cor. 9. And alſo when the revolving bodies are alſo attracting ſpheres of any condition like that above deſcribed.

PROPOSITION LXXVII. THEOREM XXXVII.

If to the ſeveral points of ſpheres there tend centripetal forces proportional to the diſtances of the points from the attracted bodies; I ſay that the compounded force with which two ſpheres attract each other mutually is as the diſtance between the centres of the ſpheres.

Case 1. Let *AEBF* (*Pl.* 22. *Fig.* 2.) be a ſphere; *S* its centre; *P* a corpuſcle attracted; *PASB* the axis of the ſphere paſſing through the centre of the corpuſcle; *EF*, *ef* two planes cutting the ſphere, and perpendicular to the axis, and equidiſtant, one on one ſide, the other on the other, from the centre of the ſphere; G and g the interſections of the planes and the axis; and H any point in the plane EF. The centripetal force of the point H upon the corpuſcle P, exerted in the direction of the line PH, is as the diſtance PH; and (by cor. 2. of the laws) the ſame exerted in the direction of the line PG, or towards the centre S, is at the length PG. Therefore the force of all the points in the plane EF (that is of that whole plane) by which the corpuſcle P is attracted towards the centre S is as the diſtance PG multiplied by the number of thoſe points, that is as the ſolid contained under that plane EF and the diſtance PG. And in like manner the force of the plane ef by which the corpuſcle P is attracted towards the centre S, is as that plane drawn into its diſtance Pg, or as the equal plane EF drawn into that diſtance Pg; and the ſum of the forces of both planes as the plane EF drawn into the ſum of the diſtances $PG + Pg$, that is as that plane drawn into twice the diſtance PS of the centre and the corpuſcle; that is, as twice the plane EF drawn into the diſtance PS, or as the ſum of the equal planes $EF + ef$ drawn into the ſame diſtance. And by a like reaſoning the forces of all the planes in the whole ſphere, equi-diſtant on each ſide from the centre of the ſphere, are as the ſum of thoſe planes drawn into the diſtance PS, that is, as the whole ſphere and the diſtance PS conjunctly. *Q. E. D.*

CASE 2. Let now the corpuſcle P attract the ſphere $AEBF$. And by the ſame reaſoning it will appear that the force with which the ſphere is attracted is as the diſtance PS. *Q. E. D.*

Case 2. Imagine another ſphere compoſed of innumerable corpuſcles *P*; and becauſe the force with which every corpuſcle is attracted is as the diſtance of the corpuſcle from the centre of the firſt ſphere, and as the ſame ſphere conjunctly, and is therefore the ſame as if it all proceeded from a ſingle corpuſcle ſituate in the centre of the ſphere; the entire force with which all the corpuſcles in the ſecond ſphere are attracted, that is, with which that whole ſphere is attracted, will be the ſame as if that ſphere were attracted by a force iſſuing from a ſingle corpuſcle in the centre of the firſt ſphere; and is therefore proportional to the diſtance between the centres of the ſpheres. *Q. E. D.*

Case 4. Let the ſpheres attract each other mutually, and the force will be doubled, but the proportion will remain. *Q. E. D.*

Case 5. Let the corpuſcle *p* be placed within the ſphere *AEBF*; (*Fig.* 3.) and becauſe the force of the plane *ef* upon the corpuſcle is as the ſolid contained under that plane and the diſtance *pg*; and the contrary force of the plane *EF* as the ſolid contained under that plane and the diſtance *pG*; the force compounded of both will be as the difference of the ſolids, that is as the ſum of the equal planes drawn into half the difference of the diſtances, that is, as that ſum drawn into *pS*, the diſtance of the corpuſcle from the centre of the ſphere. And by a like reaſoning, the attraction of all the planes *EF*, *ef* throughout the whole ſphere, that is, the attraction of the whole ſphere, is conjunctly as the ſum of all the planes, or as the whole ſphere, and as *pS* the diſtance of the corpuſcle from the centre of the ſphere. *Q. F. D.*

Case 6. And if there be compoſed a new ſphere out of innumerable corpuſcles ſuch as *p*, ſituate within the firſt ſphere *AEBF*; it may be proved as before that the attraction whether, ſingle of one ſphere towards the other, or mutual of both towards each other, will be as the diſtance *pS* of the centres. *Q. E. D.*

Proposition LXXVIII. Theorem XXXVIII.

If ſpheres in the progreſs from the centre to the circumference be however diſſimilar and unequable, but ſimilar on every ſide round about at all given diſtances from the centre; and the attractive force of every point be as the diſtance of the attracted body; I ſay that the entire force with which two ſpheres of this kind attract each other mutually is proportional to the diſtance between the centres of the ſpheres.

This is demonſtrated from the foregoing propoſition in the ſame manner as the 76th propoſition was demonſtrated from the 75th.

Cor. Thoſe things that were above demonſtrated in prop. 10. and 64. of the motion of bodies round the centres of conic ſections, take place when all the attractions are made by the force of ſphærical bodies of the condition above de-

deſcribed, and the attracted bodies are ſpheres of the ſame kind.

SCHOLIUM.

I have now explained the two principal caſes of attractions; to wit, when the centripetal forces decreaſe in a duplicate ratio of the diſtances, or increaſe in a ſimple ratio of the diſtances; cauſing the bodies in both caſes to revolve in conic ſections, and compoſing ſphærical bodies whoſe centripetal forces obſerve the ſame law of increaſe or decreaſe in the receſs from the centre as the forces of the particles themſelves do; which is very remarkable. It would be tedious to run over the other caſes, whoſe concluſions are leſs elegant and important, ſo particularly as I have done theſe. I chuſe rather to comprehend and determine them all by one general method as follows.

LEMMA XXIX.

If about the centre S (Pl. 22. Fig. 4.) *there be deſcribed any circle as* AEB, *and about the centre* P *there be alſo deſcribed two circles* EF, ef, *cutting the firſt in* E *and* e, *and the line* PS *in* F *and* f; *and there be let fall to* PS *the perpendiculars* ED, ed; *I ſay that if the diſtance of the arcs* EF, ef *be ſuppoſed to be infinitely diminiſhed, the laſt ratio of the evaneſcent*

evaneſcent line Dd *to the evaneſcent line* Ff *is the ſame as that of the line* PE *to the line* PS.

For if the line *Pe* cut the arc *EF* in *q*; and the right line *Ee*, which coincides with the evaneſcent arc *Ee*, be produced and meet the right line *PS* in *T*; and there be let fall from *S* to *PE* the perpendicular *SG*; then becauſe of the like triangles *DTE*, *dTe*, *DES*; it will be as *Dd* to *Ee* ſo *DT* to *TE*, or *DE* to *ES*; and becauſe the triangles *Eeq*, *ESG* (by lem. 8. and cor. 3. lem. 7.) are ſimilar, it will be as *Ee* to *eq* or *Ff* ſo *ES* to *SG*; and *ex æquo*, as *Dd* to *Ff* ſo *DE* to *SG*; that is (becauſe of the ſimilar triangles *PDE*, *PGS*) ſo is *PE* to *PS*. *Q. E. D.*

PROPOSITION LXXIX. THEOREM XXXIX.

Suppoſe a ſuperficies as EFfe (Pl. 22. Fig. 5.) *to have its breadth infinitely diminiſhed, and to be juſt vaniſhing; and that the ſame ſuperficies by its revolution round the axis* PS *deſcribes a ſphærical concavo-convex ſolid, to the ſeveral equal particles of which there tend equal centripetal forces; I ſay that the force with which that ſolid attracts a corpuſcle ſituate in* P, *is in a ratio compounded of the ratio of the ſolid* $DE^2 \times Ff$ *and the ratio of the force with* *which the given particle in the place* Ff *would attract the ſame corpuſcle.*

For if we conſider firſt the force of the ſphærical ſuperficies *FE* which is generated by the revolution of the arc *FE*, and is cut any where, as in *r*, by the line *de*; the annular part of the ſuperficies generated by the revolution of the arc *rE* will be as the lineola *Dd*, the radius of the ſphere *PE* remaining the ſame; as *Archimedes* has demonſtrated in his book of the ſphere and cylinder. And the force of this ſuperficies exerted in the direction of the lines *PE* or *Pr* ſituate all round in the conical ſuperficies, will be as this annular ſuperficies it ſelf; that is as the lineola *Dd*, or which is the ſame as the rectangle under the given radius *PE* of the ſphere and the lineola *Dd*; but that force, exerted in the direction of the line *PS* tending to the centre *S*, will be leſs in the ratio of *PD* to *PE*, and therefore will be as $PD \times Dd$. Suppoſe now the line *DF* to be divided into innumerable little equal particles, each of which call *Dd*; and then the ſuperficies *FE* will be divided into ſo many equal annuli, whoſe forces will be as the ſum of all the rectangles $PD \times Dd$, that is, as $\frac{1}{2}PF^2 - \frac{1}{2}PD^2$, and therefore as DE^2. Let now the ſuperficies *FE* be drawn into the altitude *Ff*; and the force of the ſolid *EFfe* exerted upon the corpuſcle *P* will be as $DE^2 \times Ff$; that is, if the force be given which any given particle as *Ff* exerts upon the corpuſcle *P* at the diſtance *PF*. But if that force be not given, the force of the ſolid *EFfe* will be as the ſolid $DE^2 \times Ff$ and that force not given, conjunctly. *Q. E. D.*

PRO-

PROPOSITION LXXX. THEOREM XL.

If to the ſeveral equal parts of a ſphere ABE, (Pl. 22. Fig. 6.) *deſcribed about the centre* S, *there tend equal centripetal forces; and from the ſeveral points* D *in the axis of the ſphere* AB *in which a corpuſcle, as* P, *is placed, there be erected the perpendiculars* DE *meeting the ſphere in* E, *and if in thoſe perpendiculars the lengths* DN *be taken as the quantity* $\frac{DE^2 \times PS}{PE}$ *and as the force which a particle of the ſphere ſituate in the axis exerts at the diſtance* PE *upon the corpuſcle* P, *conjunctly; I ſay that the whole force with which the corpuſcle* P *is attracted towards the ſphere is as the area* ANB, *comprehended under the axis of the ſphere* AB, *and the curve line* ANB, *the locus of the point* N.

For ſuppoſing the conſtruction in the laſt lemma and theorem to ſtand, conceive the axis of the ſphere *AB* to be divided into innumerable equal particles *Dd*, and the whole ſphere to be divided into ſo many ſphærical concavo-convex laminæ *EFfe*;

and erect the perpendicular dn. By the laſt theorem the force with which the lamina $EFfe$ attracts the corpuſcle P, is as $DE^2 \times Ff$ and the force of one particle exerted at the diſtance PE or PF, conjunctly. But (by the laſt lemma) Dd is to Ff as PE to PS, and therefore Ff is equal to $\frac{PS \times Dd}{PE}$; and $DE^2 \times Ff$ is equal to $Dd \times \frac{DE^2 \times PS}{PE}$; and therefore the force of the lamina $EFfe$ is as $Dd \times \frac{DE^2 \times PS}{PE}$ and the force of a particle exerted at the diſtance PF conjunctly, that is, by the ſuppoſition, as $DN \times Dd$, or as the evaneſcent area $DNnd$. Therefore the forces of all the laminæ exerted upon the corpuſcle P, are as all the area's $DNnd$, that is, the whole force of the ſphere will be as the whole area ANB. *Q. E. D.*

Cor. 1. Hence if the centripetal force tending to the ſeveral particles remain always the ſame at all diſtances, and DN be made as $\frac{DE^2 \times PS}{PE}$; the whole force with which the corpuſcle is attracted by the ſphere is as the area ANB.

Cor. 2. If the centripetal force of the particles be reciprocally as the diſtance of the corpuſcle attracted by it, and DN be made as $\frac{DE^2 \times PS}{PE^2}$; the force with which the corpuſcle P is attracted by the whole ſphere will be as the area ANB.

Cor. 3. If the centripetal force of the particles be reciprocally as the cube of the diſtance of the corpuſcle attracted by it, and DN be made as

Plate XXII. Vol. I. P. 282.

Fig. 1. p. 272.

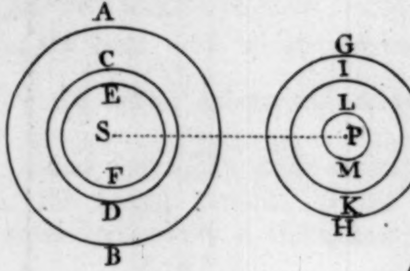


Fig. 2. p. 274

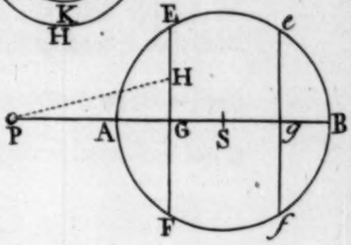


Fig. 3. p. 276.

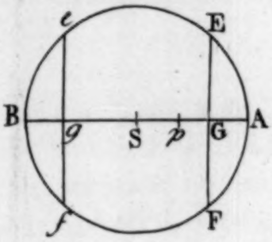


Fig. 4. p. 278.

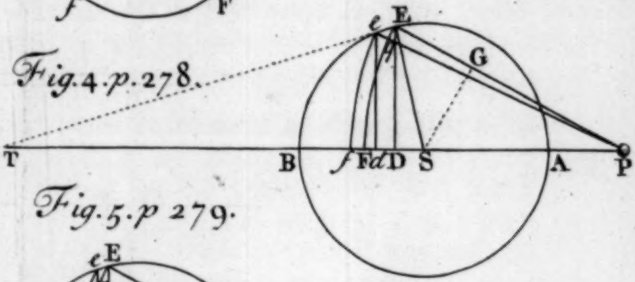


Fig. 5. p 279.

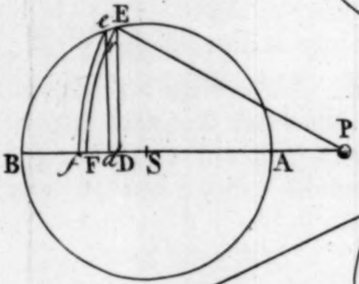
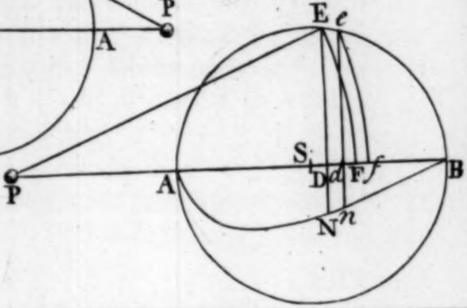


Fig. 6. p. 281.



$\frac{DE^2\times PS}{PE^4}$; the force with which the corpuſcle is attracted by the whole ſphere will be as the area *ANB*.

COR. 4. And univerſally if the centripetal force tending to the ſeveral particles of the ſphere be ſuppoſed to be reciprocally as the quantity V; and *DN* be made as $\frac{DE^2\times PS}{PE\times V}$; the force with which a corpuſcle is attracted by the whole ſphere will be as the area *ANB*.

PROPOSITION LXXXI. PROBLEM XLI.

The things remaining as above it is required to meaſure the area ANB. (Pl. 23. Fig. 1.)

From the point *P* let there be drawn the right line *PH* touching the ſphere in *H*; and to the axis *PAB* letting fall the perpendicular *HI*, biſect *PI* in *L*; and (by prop. 12. book 2. elem.) PE^2 is equal to PS^2+SE^2+2PSD. But becauſe the triangles *SPH*, *SHI* are like, SE^2 or SH^2 is equal to the rectangle *PSI*. Therefore PE^2 is equal to the rectangle contained under *PS* and $PS+SI+2SD$; that is under *PS* and $2LS+2SD$; that is under *PS* and $2LD$. Moreover DE^2 is equal to SE^2-SD^2, or $SE^2-LS^2+2SLD-LD^2$, that is, $2SLD-LD^2-ALB$. For LS^2-SE^2 or LS^2-SA^2 (by prop. 6. book 2. elem.) is equal to the rectangle *ALB*. Therefore if inſtead of DE^2 we write $2SLD-LD^2-ALB$, the quantity $\frac{DE^2\times PS}{PE\times V}$ which (by cor. 4. of the foregoing prop.) is as the length of the ordinate DN, will now reſolve it ſelf into three parts $\frac{2SLD\times PS}{PE\times V} - \frac{LD^2\times PS}{PE\times V} - \frac{ALB\times PS}{PE\times V}$; where if inſtead of V we write the inverſe ratio of the centripetal force, and inſtead of PE the mean proportional between PS and $2LD$; thoſe three parts will become ordinates to ſo many curve lines, whoſe areas are diſcovered by the common methods. *Q. F. D.*

Exam. 1. If the centripetal force tending to the ſeveral particles of the ſphere be reciprocally as the diſtance; inſtead of V write PE the diſtance; then $2PS\times LD$ for PE^2; and DN will become as $SL - \frac{1}{2}LD - \frac{ALB}{2LD}$. Suppoſe DN equal to its double $2SL - LD - \frac{ALB}{LD}$; and $2SL$ the given part of the ordinate drawn into the length AB will deſcribe the rectangular area $2SL\times AB$; and the indefinite part LD, drawn perpendicularly into the ſame length with a continued motion, in ſuch ſort as in its motion one way or another it may either by increaſing or decreaſing remain always equal to the length LD, will deſcribe the area $\frac{LB^2-LA^2}{2}$ that is, the area $SL\times AB$; which taken from the former area $2SL\times AB$ leaves the area $SL\times AB$. But the third part $\frac{ALB}{LD}$, drawn after the ſame manner with a continued motion perpendicularly into the ſame length, will deſcribe the area of an hyperbola, which ſubducted from the area $SL \times AB$ will leave ANB the area ſought. Whence ariſes this conſtruction of the problem. At the points L, A, B (*Fig.* 2.) erect the perpendiculars Ll, Aa, Bb; making Aa equal to LB, and Bb equal to LA. Making Ll and LB aſymptotes, deſcribe through the points a, b, the hyperbolic curve ab. And the chord ba being drawn will incloſe the area aba equal to the area ſought ANB.

EXAM. 2. If the centripetal force tending to the ſeveral particles of the ſphere be reciprocally as the cube of the diſtance, or (which is the ſame thing) as that cube applied to any given plane; write $\frac{PE^3}{2AS^2}$ for V, and $2PS \times LD$ for PE^2; and DN will become as $\frac{SL \times AS^2}{PS \times LD} - \frac{AS^2}{2PS} - \frac{ALB \times AS^2}{2PS \times LD^2}$ that is (becauſe PS, AS, SI are continually proportional) as $\frac{LSI}{LD} - \frac{1}{2}SI - \frac{ALB \times SI}{2LD^2}$. If we draw then theſe three parts into the length AB, the firſt $\frac{LSI}{LD}$ will generate the area of an hyperbola; the ſecond $\frac{1}{2}SI$ the area $\frac{1}{2}AB \times SI$; the third $\frac{ALB \times SI}{2LD^2}$ the area $\frac{ALB \times SI}{2LA} - \frac{ALB \times SI}{2LB}$ that is, $\frac{1}{2}AB \times SI$. From the firſt ſubduct the ſum of the ſecond and third, and there will remain ANB the area ſought. Whence ariſes this conſtruction of the problem. At the points L, A, *L*, *A*, *S*, *B*, (*Fig.* 3.) erect the perpendiculars *Ll*, *Aa*, *Ss*, *Bb*, of which ſuppoſe *Ss* equal to *SI*, and through the point *s*, to the aſymptotes *Ll*, *LB*, deſcribe the hyperbola *asb* meeting the perpendiculars *Aa*, *Bb*, in *a* and *b*; and the rectangle $2ASI$, ſubducted from the hyperbolic area *AasbB*, will leave *ANB* the area ſought.

Exam. 3. If the centripetal force tending to the ſeveral particles of the ſpheres decreaſe in a quadruplicate ratio of the diſtance from the particles; write $\frac{PE^4}{2AS^3}$ for V, then $\sqrt{2PS + LD}$ for PE, and DN will become as $\frac{SI^2 \times SL}{\sqrt{2SI}} \times \frac{1}{\sqrt{LD^3}} - \frac{SI^2}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD}} - \frac{SI^2 \times ALB}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD^5}}$. Theſe three parts drawn into the length AB, produce ſo many areas, *viz.* $\frac{2SI^2 \times SL.}{\sqrt{2SI}}$ into $\overline{\frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}}$; $\frac{SI^2}{\sqrt{2SI}}$ into $\overline{\sqrt{LB} - \sqrt{LA}}$; and $\frac{SI^2 \times ALB}{3\sqrt{2SI}}$ into $\overline{\frac{1}{\sqrt{LA^3}} - \frac{1}{\sqrt{LB^3}}}$. And theſe after due reduction come forth $\frac{2SI^2 \times SL}{LI}$, SI^2, and $SI^2 + \frac{2SI^3}{3LI}$. And theſe by ſubducting the laſt from the firſt, become $\frac{4SI^3}{3LI}$. Therefore the entire force with which the corpuſcle *P* is attracted towards the centre of the ſphere is as $\frac{SI^3}{PI}$, that is reciprocally as $PS^3 \times PI$. *Q. E. I.*

By the ſame method one may determine the attraction of a corpuſcle ſituate within the ſphere, but more expeditiouſly by the following theorem.

PROPOSITION LXXXII. THEOREM XLI.

In a ſphere deſcribed about the centre S (Pl. 23. Fig. 4.) *with the interval* SA, *if there be taken* SI, SA, SP *continually proportional; I ſay that the attraction of a corpuſcle within the ſphere in any place* I, *is to its attraction without the ſphere in the place* P, *in a ratio compounded of the ſubduplicate ratio of* IS, PS *the diſtances from the centre, and the ſubduplicate ratio of the centripetal forces tending to the centre in thoſe places* P *and* I.

As if the centripetal forces of the particles of the ſphere be reciprocally as the diſtances of the corpuſcle attracted by them; the force with which the corpuſcle ſituate in *I* is attracted by the entire ſphere, will be to the force with which it is attracted in *P*, in a ratio compounded of the ſubduplicate ratio of the diſtance *SI* to the diſtance *SP*, and the ſubduplicate ratio of the centripetal force in the place *I* ariſing from any particle in the centre, to the centripetal force in the place *P* ariſing from the ſame particle in the centre, that is. in the ſubduplicate ratio of the diſtances SI, SP to each other reciprocally. Theſe two ſubduplicate ratio's compoſe the ratio of equality, and therefore the attractions in I and P produced by the whole ſphere are equal. By the like calculation if the forces of the particles of the ſphere are reciprocally in a duplicate ratio of the diſtance, it will be found that the attraction in I is to the attraction in P as the diſtance SP to the ſemi-diameter SA of the ſphere. If thoſe forces are reciprocally in a triplicate ratio of the diſtances, the attractions in I and P will be to each other as SP^2 to SA^2; if in a quadruplicate ratio, as SP^3 to SA^3. Therefore ſince the attraction in P was found in this laſt caſe to be reciprocally as $PS^3 \times PI$, the attraction in I will be reciprocally as $SA^3 \times PI$, that is, becauſe SA^3 is given, reciprocally as PI. And the progreſſion is the ſame in infinitum. The demonſtration of this theorem is as follows.

The things remaining as above conſtructed, and a corpuſcle being in any place P, the ordinate DN was found to be as $\frac{DE^2 \times PS}{PE \times V}$. Therefore if IE be drawn, that ordinate for any other place of the corpuſcle as I, will become (*mutatis mutandis*) as $\frac{DE^2 \times IS}{IE \times V}$. Suppoſe the centripetal forces flowing from any point of the ſphere as E, to be to each other at the diſtances IE and PE, as PE^n to IE^n (where the number n denotes the index of the powers of PE and IE) and thoſe ordinates will become as $\frac{DE^2 \times PS}{PE \times PE^n}$ and $\frac{DE^2 \times IS}{IE \times IE^n}$ whoſe ratio to each other is as $PS \times IE \times IE^n$ to $IS \times PE \times PE^n$. Becauſe *SI*, *SE*, *SP* are in continued proportion, the triangles *SPE*, *SEI* are alike; and thence *IE* is to *PE* as *IS* to *SE* or *SA*. For the ratio of *IE* to *PE* write the ratio of *IS* to *SA*; and the ratio of the ordinates becomes that of $PS \times IE^n$ to $SA \times PE^n$. But the ratio of *PS* to *SA* is ſubduplicate of that of the diſtances *PS*, *SI*; and the ratio of IE^n to PE^n (becauſe *IE* is to *PE* as *IS* to *SA*) is ſubduplicate of that of the forces at the diſtances *PS*, *IS*. Therefore the ordinates, and conſequently the areas which the ordinates deſcribe, and the attractions proportional to them, are in a ratio compounded of thoſe ſubduplicate ratio's. *Q. E. D.*

PROPOSITION LXXXIII. PROBLEM XLII.

To find the force with which a corpuſcle placed in the centre of a ſphere is attracted towards any ſegment of that ſphere whatſoever.

Let *P* (*Pl.* 23. *Fig.* 5.) be a body in the centre of that ſphere, and *RBSD* a ſegment thereof contained under the plane *RDS* and the ſphærical ſuperficies *RBS*. Let *DB* be cut in *F* by a ſphærical ſuperficies *EFG* deſcribed from the centre *P*, and let the ſegment be divided into the parts *BREFGS*, *FEDG*. Let us ſuppoſe that ſegment to be not a purely mathematical, but a phyſical ſuperficies, having ſome, but a perfectly inconſiderable thickneſs. Let that thickneſs be called O and (by what *Archimedes* has demonſtrated) that ſuperficies will be as $PF \times DF \times O$. Let us ſuppoſe beſides the attractive forces of the particles of the ſphere to be reciprocally as that power of the diſtances, of which *n* is index; and the force with which the ſuperficies *EFG* attracts the body *P*, will be (by prop. 79.) as $\frac{DE^2 \times O}{PF^n}$, that is, as $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^2 \times O}{PF^n}$. Let the perpendicular *FN* drawn into O be proportional to this quantity; and the curvilinear area *BDI*, which the ordinate *FN*, drawn through the length *DB* with a continued motion will deſcribe, will be as the whole force with which the whole ſegment *RBSD* attracts the body *P*. *Q. E. I.*

PROPOSITION LXXXIV. PROBLEM XLIII.

To find the force with which a corpuſcle, placed without the centre of a ſphere in the axis of any ſegment, is attracted by that ſegment.

Let the body *P* placed in the axis *ADB* of the ſegment *EBK* (*Pl.* 23. *Fig.* 6.) be attracted by that ſegment. About the centre *P* with the interval *PE* let the ſphærical ſuperficies *EFK* be deſcribed; and let it divide the ſegment into two parts *EBKFE* and *EFKDE*. Find the force of the firſt of thoſe parts by prop. 81. and the force of the latter part by prop. 83. and the ſum of the forces will be the force of the whole ſegment *EBKDE*. *Q. E. I.*

SCHOLIUM.

The attractions of ſphærical bodies being now explained, it comes next in order to treat of the laws of attraction in other bodies conſiſting in like manner of attractive particles; but to treat of them particularly is not neceſſary to my deſign. It will be ſufficient to ſubjoin ſome general propoſitions relating to the forces of ſuch bodies, and the motions thence ariſing, becauſe the knowledge of theſe will be of ſome little uſe in philoſophical enquiries.

SECTION XIII.

Of the attractive forces of bodies which are not of a ſphærical figure.

PROPOSITION LXXXV. THEOREM XLII.

If a body be attracted by another, and its attraction be vaſtly ſtronger when it is contiguous to the attracting body, than when they are ſeparated from one another by a very ſmall interval; the forces of the particles of the attracting body decreaſe, in the receſs of the body attracted, in more than a duplicate ratio of the diſtance of the particles.

For if the forces decreaſe in a duplicate ratio of the diſtances from the particles, the attraction towards a ſphærical body, being (by prop. 74.) reciprocally as the ſquare of the diſtance of the attracted body from the centre of the ſphere, will not be ſenſibly increaſed by the contact, and it will be ſtill leſs increaſed by it, if the attraction, in the receſs of the body attracted, decreaſes in a ſtill leſs proportion. The propoſition therefore is evident concerning attractive ſpheres. And the caſe is the ſame of concave ſphærical orbs attracting external bodies. And much more does it appear in orbs that attract bodies placed within them, becauſe there the attractions diffuſed through the cavities of thoſe orbs are (by prop. 70.) deſtroyed by contrary attractions, and therefore have no effect even in the place of contact. Now if from theſe ſpheres and ſphærical orbs we take away any parts remote from the place of contact, and add new parts any where at pleaſure; we may change the figures of the attractive bodies at pleaſure, but the parts added or taken away, being remote from the place of contact, will cauſe no remarkable exceſs of the attraction ariſing from the contact of the two bodies. Therefore the propoſition holds good in bodies of all figures. *Q. E. D.*

PROPOSITION LXXXVI. THEOREM XLIII.

If the forces of the particles of which an attractive body is compoſed, decreaſe, in the receſs of the attracted body, in a triplicate or more than a triplicate ratio of the diſtance from the particles; the attraction will be *vaſtly ſtronger in the point of contact than when the attracting and attracted bodies are ſeparated from each other though by never ſo ſmall an interval.*

For that the attraction is infinitely increaſed when the attracted corpuſcle comes to touch an attracting ſphere of this kind appears by the ſolution of problem 41. exhibited in the ſecond and third examples. The ſame will alſo appear (by comparing thoſe examples and theorem 41. together) of attractions of bodies made towards concavo-convex orbs, whether the attracted bodies be placed without the orbs, or in the cavities within them. And by adding to or taking from thoſe ſpheres and orbs, any attractive matter any where without the place of contact, ſo that the attractive bodies may receive any aſſigned figure, the propoſition will hold good of all bodies univerſally. *Q. E. D.*

PROPOSITION LXXXVII. THEOREM XLIV.

If two bodies ſimilar to each other, and conſiſting of matter equally attractive, attract ſeparately two corpuſcles proportional to thoſe bodies, and in a like ſituation to them; the accelerative attractions of the corpuſcles towards the entire bodies will be as the

the accelerative attractions of the corpuſcles towards particles of the bodies proportional to the wholes, and alike ſituated in them.

For if the bodies are divided into particles proportional to the wholes and alike ſituated in them, it will be, as the attraction towards any particle of one of the bodies to the attraction towards the correſpondent particle in the other body, ſo are the attractions towards the ſeveral particles of the firſt body to the attractions towards the ſeveral correſpondent particles of the other body; and by compoſition, ſo is the attraction towards the firſt whole body to the attraction towards the ſecond whole body. *Q. E. D.*

Cor. 1. Therefore, if as the diſtances of the corpuſcles attracted increaſe, the attractive forces of the particles decreaſe in the ratio of any power of the diſtances; the accelerative attractions towards the whole bodies will be as the bodies directly and thoſe powers of the diſtances inverſely. As if the forces of the particles decreaſe in a duplicate ratio of the diſtances from the corpuſcles attracted, and the bodies are as A^3 and B^3, and therefore both the cubic ſides of the bodies, and the diſtance of the attracted corpuſcles from the bodies are as A and B; the accelerative attractions towards the bodies will be as $\frac{A^3}{A^2}$ and $\frac{B^3}{B^2}$, that is, as A and B the cubic ſides of thoſe bodies. If the forces of the particles decreaſe in a triplicate ratio of the diſtances from the attracted corpuſcles; the accelerative attractions towards the whole bodies will be as $\frac{A^3}{A^3}$ and $\frac{B^3}{B^3}$, that is, equal. If the forces decreaſe in a quadruplicate ratio; the attractions towards the bodies will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$ that is, reciprocally as the cubic ſides A and B. And ſo in other caſes.

COR. 2. Hence on the other hand, from the forces with which like bodies attract corpuſcles ſimilarly ſituated, may be collected the ratio of the decreaſe of the attractive forces of the particles as the attracted corpuſcle recedes from them; if ſo be that decreaſe is directly or inverſely in any ratio of the diſtances.

PROPOSITION LXXXVIII. THEOREM XLV.

If the attractive forces of the equal particles of any body be as the diſtance of the places from the particles, the force of the whole body will tend to its centre of gravity; and will be the ſame with the force of a globe, conſiſting of ſimilar and equal matter, and having its centre in the centre of gravity.

Let the particles A, B, (*Pl.* 23. *Fig.* 7.) of the body $RSTV$, attract any corpuſcle Z with forces which, ſuppoſing the particles to be equal between themſelves, are as the diſtances AZ, BZ; but if they are ſuppoſed unequal, are as thoſe particles and their diſtances AZ, BZ conjunctly, or (if I may ſo ſpeak) as thoſe particles drawn into their diſtances AZ, BZ reſpectively. And let thoſe forces be expreſſed by the contents under $A \times AZ$, and $B \times BZ$. Join AB, and let it be cut in G, ſo that AG may be to BG as the particle B to the particle A; and G will be the common centre of gravity of the particles A and B. The force $A \times AZ$ will (by cor. 2. of the laws) be reſolved into the forces $A \times GZ$ and $A \times AG$; and the force $B \times BZ$ into the forces $B \times GZ$ and $B \times BG$. Now the forces $A \times AG$ and $B \times BG$, becauſe A is proportional to B, and BG to AG, are equal; and therefore having contrary directions deſtroy one other. There remain then the forces $A \times GZ$ and $B \times GZ$. Theſe tend from Z towards the centre G, and compoſe the force $\overline{A+B} \times GZ$; that is the ſame force as if the attractive particles A and B were placed in their common centre of gravity G, compoſing there a little globe.

By the ſame reaſoning if there be added a third particle C, and the force of it be compounded with the force $\overline{A+B} \times GZ$ tending to the centre G; the force thence ariſing will tend to the common centre of gravity of that globe in G and of the particle C; that is, to the common centre of gravity of the three particles A, B, C; and will be the ſame as if that globe and the particle C were placed in that common centre compoſing a greater globe there. And ſo we may go on *in infinitum*. Therefore the whole force of all the particles of any body whatever *RSTV*, is the ſame as if that body, without removing its centre of gravity, were to put on the ſorm of a globe. *Q. E. D.*

Cor. Hence the motion of the attracted body *Z* will be the ſame, as if the attracting body *RSTV* were ſphærical; and therefore if that attracting body be either at reſt, or proceed uniformly in a right line; the body attracted will move in an ellipſis having its centre in the centre of gravity of the attracting body.

PROPOSITION LXXXIX. THEOREM XLVI.

If there be ſeveral bodies conſiſting of equal particles whoſe forces are as the diſtances of the places from each; the force compounded of all the forces by which any corpuſcle is attracted, will tend to the common centre of gravity of the attracting bodies; and will be the ſame as if thoſe attracting bodies, preſerving their common centre of gravity, ſhould unite there, and be formed into a globe.

This is demonſtrated after the ſame manner as the foregoing propoſition.

Cor. Therefore the motion of the attracted body will be the ſame as if the attracting bodies, preſerving their common centre of gravity, ſhould unite

Plate XXIII. *Vol.* I. *P.* 298.

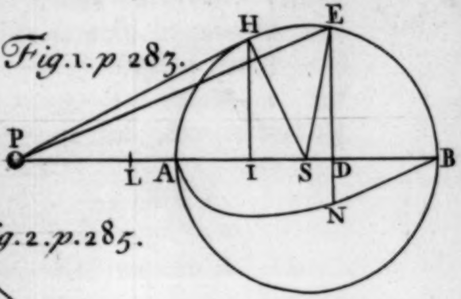


Fig.1.p 283.

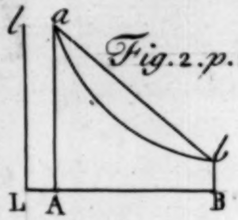


Fig.2.p.285.

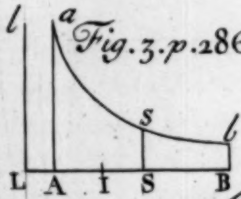


Fig.3.p.286.

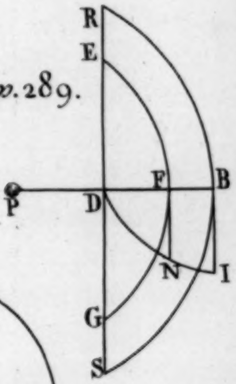


Fig.5.p.289.

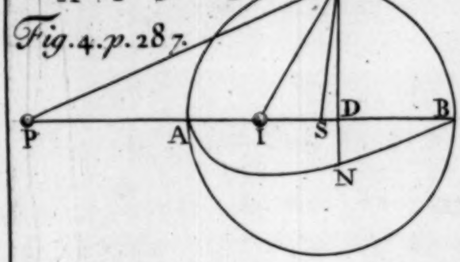


Fig.4.p.287.

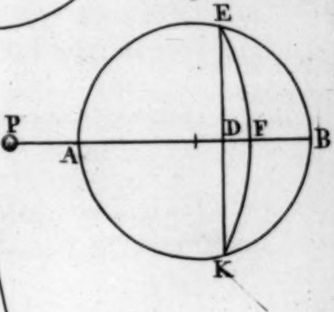


Fig.6.p.290.

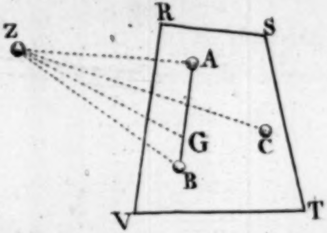


Fig.7.p.296.

unite there, and be formed into a globe. And therefore if the common centre of gravity of the attracting bodies be either at reſt, or proceeds uniformly in a right line; the attracted body will move in an ellipſis having its centre in the common centre of gravity of the attracting bodies.

PROPOSITION XC. PROBLEM XLIV.

If to the ſeveral points of any circle there tend equal centripetal forces, increaſing or decreaſing in any ratio of the diſtances; it is required to find the force with which a corpuſcle is attracted, that is ſituate any where in a right line which ſtands at right angles to the plane of the circle at its centre.

Suppoſe a circle to be deſcribed about the centre *A* (*Pl.* 24. *Fig.* 1.) with any interval *AD* in a plane to which the right line *AP* is perpendicular; and let it be required to find the force with which a corpuſcle *P* is attracted towards the ſame. From any point *E* of the circle, to the attracted corpuſcle *P*, let there be drawn the right line *PE*. In the right line *PA* take *PF* equal to *PE*, and make a perpendicular *FK*, erected at *F*, to be as the force with which the point *E* attracts the corpuſcle *P*. And let the curve line *IKL* be the locus of the point *K*. Let that curve meet the plane of the circle in *L*. In *PA* take *PH* equal to *PD*, and erect the perpendicular *HI* meeting that curve in *I*; and the attraction of the corpuſcle *P* towards the circle will be as the area *AHIL* drawn into the altitude *AP*. *Q. E. I.*

For let there be taken in *AE* a very ſmall line *Ee*. Join *Pe*, and in *PE*, *PA* take *PC*, *Pf* equal to *Pe*. And becauſe the force with which any point *E* of the annulus deſcribed about the centre *A* with the interval *AE* in the aforeſaid plane, attracts to it ſelf the body *P*, is ſuppoſed to be as *FK*; and therefore the force with which that point attracts the body *P* towards *A* is as $\frac{AP \times FK}{PE}$; and the force with which the whole annulus attracts the body *P* towards *A*, is as the annulus and $\frac{AP \times FK}{PE}$ conjunctly; and that annulus alſo is as the rectangle under the radius *AE* and the breadth *Ee*, and this rectangle (becauſe *PE* and *AE*, *Ee* and *CE* are proportional) is equal to the rectangle $PE \times CE$ or $PE \times Ff$; the force with which that annulus attracts the body *P* towards *A*, will be as $PE \times Ff$ and $\frac{AP \times FK}{PE}$ conjunctly; that is as the content under $Ff \times FK \times AP$, or as the area *FKkf* drawn into *AP*. And therefore the ſum of the forces with which all the annuli, in the circle deſcribed about the centre *A* with the interval *AD*, attract the body *P* towards *A*, is as the whole area *AHIKL* drawn into *AP*. *Q. E. D.*

COR. 1. Hence if the forces of the points decreaſe in the duplicate ratio of the diſtances, that is, if FK be as $\frac{1}{PF^2}$, and therefore the area $AHIKL$ as $\frac{1}{PA} - \frac{1}{PH}$; the attraction of the corpuſcle P towards the circle will be as $1 - \frac{PA}{PH}$; that is, as $\frac{AH}{PH}$.

COR, 2. And univerſally if the forces of the points at the diſtances D be reciprocally as any power D^n of the diſtances; that is, if FK be as $\frac{1}{D^n}$, and therefore the area $AHIKL$ as $\frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}$; the attraction of the corpuſcle P towards the circle will be as $\frac{1}{PA^{n-2}} - \frac{PA}{PH^{n-1}}$.

COR. 3. And if the diameter of the circle be increaſed in infinitum, and the number n be greater than unity; the attraction of the corpuſcle P towards the whole infinite plane will be reciprocally as PA^{n-2} becauſe the other term $\frac{PA}{PH^{n-1}}$ vaniſhes.

PRO-

PROPOSITION XCI. PROBLEM XLV.

To find the attraction of a corpuſcle ſituate in the axis of a round ſolid, to whoſe ſeveral points there tend equal centripetal forces decreaſing in any ratio of the diſtances whatſoever.

Let the corpuſcle *P* (*Pl.* 24. *Fig.* 2.) ſituate in the axis *AB* of the ſolid *DECG*, be attracted towards that ſolid. Let the ſolid be cut by any circle as *RFS*, perpendicular to the axis; and in its ſemi-diameter *FS*, in any plane *PALKB* paſſing through the axis, let there be taken (by prop. 90.) the length *FK* proportional to the force with which the corpuſcle *P* is attracted towards that circle. Let the locus of the point *K* be the curve line *LKI*, meeting the planes of the outermoſt circles *AL* and *BI* in *L* and *I*; and the attraction of the corpuſcle *P* towards the ſolid will be as the area *LABI*. *Q. E. I.*

Cor. 1. Hence if the ſolid be a cylinder deſcribed by the parallelogram *ADEB* (*Pl.* 24. *Fig.* 3.) revolved about the axis *AB*, and the centripetal forces tending to the ſeveral points be reciprocally as the ſquares of the diſtances from the points; the attraction of the corpuſcle *P* towards this cylinder will be as $AB - PE + PD$. For the ordinate *FK* (by cor. 1. prop. 90.) will be as

1—

$1-\frac{PF}{PR}$. The part 1 of this quantity, drawn into the length AB, deſcribes the area $1\times AB$; and the other part $\frac{PF}{PR}$, drawn into the length PB, deſcribes the area 1 into $\overline{PE-AD}$, (as may be eaſily ſhewn from the quadrature of the curve LKI;) and in like manner, the ſame part drawn into the length PA deſcribes the area 1 into $\overline{PD-AD}$, and drawn into AB, the difference of PB and PA, deſcribes 1 into $\overline{PE-PD}$, the difference of the areas. From the firſt content $1\times AB$ take away the laſt content 1 into $\overline{PE-PD}$, and there will remain the area $LABI$ equal to 1 into $\overline{AB-PE+PD}$. Therefore the force being proportional to this area, is as $AB-PE+PD$.

Cor. 2. Hence alſo is known the force by which a ſpheroid $AGBC$ (*Pl.* 24. *Fig.* 4.) attracts any body P ſituate externally in its axis AB. Let $NKRM$ be a conic ſection whoſe ordinate ER perpendicular to PE, may be always equal to the length of the line PD, continually drawn to the point D in which that ordinate cuts the ſpheroid. From the vertices A, B, of the ſpheriod, let there be erected to its axis AB the perpendiculars AK, BM, reſpectively equal to AP, BP, and therefore meeting the conic ſection in K and M; and join KM cutting off from it the ſegment $KMRK$. Let S be the centre of the ſpheriod, and SC its greateſt ſemi-diameter; and the force with which the ſpheroid attracts the body P, will be to the force with which a ſphere deſcribed with the diameter AB attracts the ſame body, dy, as $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ is to $\frac{AS^3}{3PS^2}$. And by a calculation founded on the ſame principles may be found the forces of the ſegments of the ſpheroid.

Cor. 3. If the corpuſcle be placed within the ſpheroid and in its axis, the attraction will be as its diſtance from the centre. This may be eaſily collected from the following reaſoning, whether the particle be in the axis or in any other given diameter. Let *AGOF* (*Pl.* 24. *Fig.* 5.) be an attracting ſpheroid, *S* its centre, and *P* the body attracted. Through the body *P* let there be drawn the ſemi-diameter *SPA*, and two right lines *DE*, *FG* meeting the ſpheroid in *D* and *E*, *F* and *G*; and let *PCM*, *HLN* be the ſuperficies of two interior ſpheroids ſimilar and concentrical to the exterior, the firſt of which paſſes through the body *P*, and cuts the right lines *DE*, *FG* in *B* and *C*; and the latter cuts the ſame right lines in *H* and *I*, *K* and *L*. Let the ſpheroids have all one common axis, and the parts of the right lines intercepted on both ſides *DP* and *BE*, *FP* and *CG*, *DH* and *IE*, *FK* and *LG* will be mutually equal; becauſe the right lines *DE*, *PB*, and *HI* are biſected in the ſame point, as are alſo the right lines *FG*, *PC* and *KL*. Conceive now *DPF*, *EPG* to repreſent oppoſite cones deſcribed with the infinitely ſmall vertical angles *DPF*, *EPG*, and the lines *DH*, *EI* to be infinitely ſmall alſo. Then the particles of the cones *DHKF*, *GLIE*, cut off by the ſpheroidical ſuperficies, by reaſon of the equality of the lines *DH* and *EI*, will be to one another as the ſquares of the diſtances from the body *P*, and will therefore attract that corpuſcle equally. And by a like reaſoning if the ſpaces *DPF, EGCB* be divided into particles by the ſuperficies of innumerable ſimilar ſpheroids concentric to the former and having one common axis, all theſe particles will equally attract on both ſides the body *P* towards contrary parts. Therefore the forces of the cone *DPF*, and of the conic ſegment *EGCB* are equal and by their contrariety deſtroy each other. And the caſe is the ſame of the forces of all the matter that lies without the interior ſpheroid *PCBM*. Therefore the body *P* is attracted by the interior ſpheroid *PCBM* alone, and therefore (by cor. 3. prop. 72.) its attraction is to the force with which the body *A* is attracted by the whole ſpheroid *AGOD*, as the diſtance *PS* to the diſtance *AS*. *Q. E. D.*

PROPOSITION XCII. PROBLEM XLVI.

An attracting body being given, it is required to find the ratio of the decreaſe of the centripetal forces tending to its ſeveral points.

The body given muſt be formed into a ſphere, a cylinder, or ſome regular figure whoſe, law of attraction anſwering to any ratio of decreaſe may be found by prop. 80. 81 and 91. Then, by experiments, the force of the attractions muſt be found at ſeveral diſtances, and the law of attraction towards the whole, made known by that means, will give the ratio of the decreaſe of the forces of the ſeveral parts; which was to be found.

PROPOSITION XCIII. THEOREM XLVII.

If a ſolid be plane on one ſide, and infinitely extended on all other ſides, and conſiſt of equal particles equally attractive, whoſe forces decreaſe, in the receſs from the ſolid, in the ratio of any power greater than the ſquare of the diſtances; and a corpuſcle placed towards either part of the plane is attracted by the force of the whole ſolid; I ſay that the attractive force of the whole ſolid, in the receſs from its plane ſuperficies, will decreaſe in the ratio of a power whoſe ſide is the diſtance of the corpuſcle from the plane, and its index leſs by 3 *than the index of the power of the diſtances.*

CASE 1. Let *LGl* (*Pl.* 24. *Fig.* 6.) be the plane by which the ſolid is terminated. Let the ſolid lie on that hand of the plane that is towards *I*, and let it be reſolved into innumerable planes *mHM*, *nIN*, *oKO*, *&c.* parallel to *GL*. And firſt let the attracted body *C* be placed without the ſolid. Let there be drawn *CGHI* perpendicular to thoſe innumerable planes, and let the attractive forces of the points of the ſolid decreaſe in the ratio of a power of the diſtances whoſe index is the number *n* not leſs than 3. Therefore (by cor. 3. prop. 90.) the force with which any plane *mHM* attracts the point *C*, is reciprocally as CH^{n-2}. In the plane *mHM* take the length *HM* reciprocally proportional to CH^{n-2}, and that force will be as *HM*. In like manner in the ſeveral planes *lGL*, *nIN*, *oKO*, *&c.* take the lengths *GL*, *IN*, *KO*, *&c.* reciprocally proportional to CG^{n-2}, CI^{n-2}, CK^{n-2}, *&c.* and the forces of thoſe planes will be as the lengths ſo taken, and therefore the ſum of the forces as the ſum of the lengths, that is, the force of the whole ſolid as the area *GLOK* produced infinitely towards *OK*. But that area (by the known methods of quadratures) is reciprocally as CG^{n-3}, and therefore the force of the whole ſolid is reciprocally as CG^{n-3}. *Q. E. D.*

CASE 2. Let the corpuſcle *C* (*Fig.* 7.) be now placed on that hand of the plane *lGL* that is within the ſolid, and take the diſtance *CK* equal to the diſtance *CG*. And the part of the ſolid *LGloKO* terminated by the parallel planes *lGL*, *oKO*, will attract the corpuſcle, ſituate in the middle, neither one way nor another, the contrary actions of the oppoſite points deſtroying one another by reaſon of their equality. Therefore the corpuſcle *C* is attracted by the force only of the ſolid ſituate beyond the plane *OK*. But this force (by caſe 1.) is reciprocally as CK^{n-3}, that is (becauſe *CG*, *CK* are equal) reciprocally as CG^{n-3}. *Q. E. D.*

COR. 1. Hence if the ſolid *LGIN* be terminated on each ſide by two infinite parallel planes *LG*, *IN*; its attractive force is known, ſubducting from the attractive force of the whole infinite ſolid *LGKO*, the attractive force of the more diſtant part *NIKO* infinitely produced towards *KO*.

Cor. 2. If the more diſtant part of this ſolid be rejected, becauſe its attraction compared with the attraction of the nearer part is inconſiderable; the attraction of that nearer part will, as the diſtance increaſes, decreaſe nearly in the ratio of the power CG^{n-3}.

Cor. 3. And hence if any finite body, plane on one ſide, attract a corpuſcle ſituate over-againſt the middle of that plane, and the diſtance between the corpuſcle and the plane compared with the dimenſions of the attracting body be extremely ſmall; and the attracting body conſiſt of homogeneous particles, whoſe attractive forces decreaſe in the ratio of any power of the diſtances greater than the quadruplicate; the attractive force of the whole body will decreaſe very nearly in the ratio of a power whoſe ſide is that very ſmall diſtance, and the index leſs by 3 than the index of the former power. This aſſertion does not hold good however of a body conſiſting of particles whoſe attractive forces decreaſe in the ratio of the triplicate power of the diſtances; becauſe in that caſe, the attraction of the remoter part of the infinite body in the ſecond corollary is always infinitely greater than the attraction of the nearer part.

SCHOLIUM.

If a body is attracted perpendicularly towards a given plane, and from the law of attraction given the motion of the body be required; the problem will be ſolved by ſeeking (by prop. 39.) the motion of the body deſcending in a right line towards that plane, and (by cor. 2. of the laws) compounding that motion with an uniform motion, performed in the

Plate XXIV. Vol. I. P. 308.

Fig. 1. p. 299.

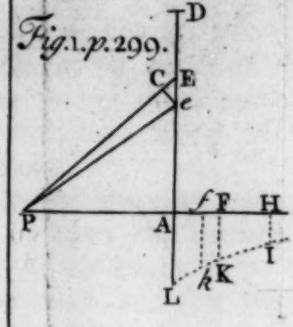


Fig. 2. p. 302.

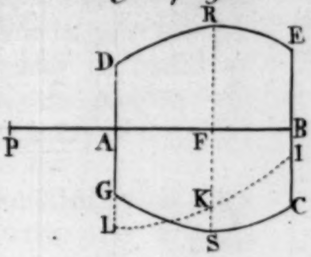


Fig. 3. ibid.

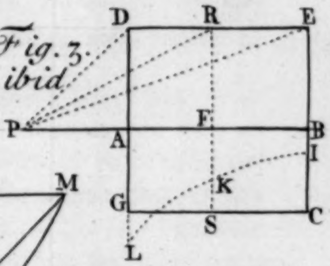


Fig. 4. p. 303.

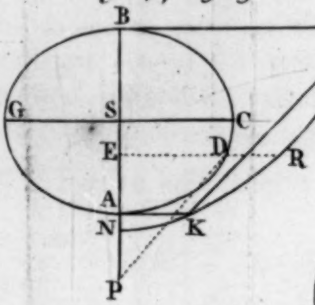


Fig. 5. p. 304.

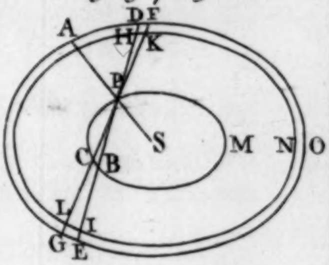


Fig. 6. p. 306.

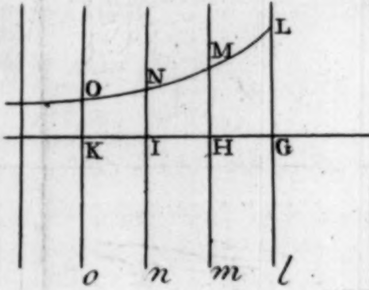
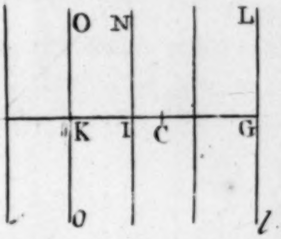


Fig. 7. p. 307.



the direction of lines parallel to that plane. And on the contrary if there be required the law of the attraction tending towards the plane in perpendicular directions, by which the body may be cauſed to move in any given curve line, the problem will be ſolved by working after the manner of the third problem.

But the operations may be contracted by reſolving the ordinates into converging ſeries. As if to a baſe A the length B be ordinately applied in any given angle, and that length be as any power of the baſe $A^{\frac{m}{n}}$; and there be ſought the force with which a body, either attracted towards the baſe or driven from it in the direction of that ordinate, may be cauſed to move in the curve line which that ordinate always deſcribes with its ſuperior extremity; I ſuppoſe the baſe to be increaſed by a very ſmall part O, and I reſolve the ordinate $\overline{A+O}|^{\frac{m}{n}}$ into an infinite ſeries $A^{\frac{m}{n}} + \frac{m}{n}OA^{\frac{m-n}{n}} + \frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$ &c. and I ſuppoſe the force proportional to the term of this ſeries in which O is of two dimenſions, that is, to the term $\frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$. Therefore the force ſought is as $\frac{mm-mn}{nn}A^{\frac{m-2n}{n}}$, or which is the ſame thing, as $\frac{mm-mn}{nn}B^{\frac{m-2n}{m}}$. As if the ordinate deſcribe a parabola, m being $=2$, and $n=1$, the force will be as the given quantity $2B^{0}$, and therefore is given. Thereſore with a given ſorce the body will move in a parabola, as *Galileo* has demonſtrated. If the ordinate deſcribe an hyperbola, m being $=0-1$, and $n=1$; the ſorce will be as $2A^{-3}$ or $2B^{3}$; and thereſore a ſorce which is as the cube of the ordinate will cauſe the body to move in an hyperbola. But leaving this kind of propoſitions, I ſhall go on to ſome others relating to motion which I have not yet touched upon.

SECTION XIV.

Of the motion of very ſmall bodies when agitated by centripetal forces tending to the ſeveral parts of any very great body.

PROPOSITION XCIV. THEOREM XLVIII.

If two ſimilar mediums be ſeparated from each other by a ſpace terminated on both ſides by parallel planes, and a body in its paſſage through that ſpace be attracted or impelled perpendicularly towards either of thoſe mediums, and not agitated or hindered by any other force; and the attraction be every where the ſame at equal diſtances from either plane, taken towards the ſame hand of the plane; I ſay that the ſine of incidence upon either plane will be to the ſine of emergence from the other plane, in a given ratio.

Case 1. Let *Aa* and *Bb* (*Pl.* 25. *Fig.* 1.) be two parallel planes, and let the body light upon the

the firſt plane Aa in the direction of the line GH, and in its whole paſſage through the intermediate ſpace let it be attracted or impelled towards the medium of incidence, and by that action let it be made to deſcribe a curve line HI, and let it emerge in the direction of the line IK. Let there be erected IM perpendicular to Bb the plane of emergence, and meeting the line of incidence GH prolonged in M, and the plane of incidence Aa in R; and let the line of emergence KI be produced and meet HM in L. About the centre L, with the interval LI, let a circle be deſcribed cutting both HM in P and Q, and MI produced in N; and firſt, if the attraction or impulſe be ſuppoſed uniform, the curve HI (by what *Galileo* has demonſtrated) be a parabola, whoſe property is, that a rectangle under its given latus rectum and the line IM is equal to the ſquare of HM; and moreover the line HM will be biſected in L. Whence if to MI there be let fall the perpendicular LO, MO, OR will be equal; and adding the equal lines ON, OI, the wholes MN, IR will be equal alſo. Therefore ſince IR is given MN is alſo given, and the rectangle NMI is to the rectangle under the latus rectum and IM, that is, to HM^2, in a given ratio. But the rectangle NMI is equal to the rectangle PMQ, that is, to the difference of the ſquares ML^2, and PL^2 or LI^2; and HM^2 hath a given ratio to its fourth part ML^2; therefore the ratio of $ML^2 - LI^2$ to ML^2 is given, and by converſion the ratio of LI^2 to ML^2, and its ſubduplicate, the ratio of LI to ML. But in every triangle as LMI, the ſines of the angles are proportional to the oppoſite ſides. Therefore the ratio of the ſine of the angle of incidence LMR

to the ſine of the angle of emergence *LIR* is given. *Q. E. D.*

CASE 2. Let now the body paſs ſucceſſively through ſeveral ſpaces terminated with parallel planes, *AabB*, *BbcC*, &c. (*Pl.* 25. *Fig.* 2.) and let it be acted on by a force which is uniform in each of them ſeparately, but different in the different ſpaces; and by what was juſt demonſtrated, the ſine of the angle of incidence on the firſt plane *Aa* is to the ſine of emergence from the ſecond plane *Bb* in a given ratio; and this ſine of incidence upon the ſecond plane *Bb* will be to the ſine of emergence from the third plane *Cc* in a given ratio; and this ſine to the ſine of emergence from the fourth plane *Dd* in a given ratio; and ſo on in infinitum; and by equality, the ſine of incidence on the firſt plane to the ſine of emergence from the laſt plane in a given ratio. Let now the intervals of the planes be diminiſhed, and their number be infinitely increaſed, ſo that the action of attraction or impulſe, exerted according to any aſſigned law, may become continual, and the ratio of the ſine of incidence on the firſt plane to the ſine of emergence from the laſt plane being all along given, will be given then alſo. *Q. E. D.*

PROPOSITION XCV. THEOREM XLIX.

The ſame things being ſuppoſed, I ſay that the velocity of the body before its incidence is to its velocity after emergence as the ſine of emergence to the ſine of incidence.

Make *AH* and *Id* equal (*Pl.* 25. *Fig.* 3.) and erect the perpendiculars *AG*, *dK* meeting the lines of incidence and emergence *GH*, *IK* in *G* and *K*. In *GH* take *TH* equal to *IK*, and to the plane *Aa* let fall a perpendicular *Tv*. And (by cor. 2. of the laws of motion) let the motion of the body be reſolved into two, one perpendicular to the planes *Aa*, *Bb*, *Cc*, &c. and another parallel to them. The force of attraction or impulſe, acting in directions perpendicular to thoſe planes, does not at all alter the motion in parallel directions; and therefore the body proceeding with this motion will in equal times go through thoſe equal parallel intervals that lie between the line *AG* and the point *H*, and between the point *I* and the line *dK*; that is, they will deſcribe the lines *GH*, *IK* in equal times. Therefore the velocity before incidence is to the velocity after emergence as *GH* to *IK* or *TH*, that is as *AH* or *Id* to *vH*, that is (ſuppoſing *TH* or *IK* radius) as the ſine of emergence to the ſine of incidence. *Q. E. D.*

PROPOSITION XCVI. THEOREM L.

The ſame things being ſuppoſed, and that the motion before incidence is ſwifter than afterwards; I ſay that if the line of incidence be inclined continually, the body will be at laſt reflected, and the angle of reflexion will be equal to the angle of incidence.

For conceive the body paſſing between the parallel planes *Aa*, *Bb*, *Cc*, &c. (*Pl.* 25. *Fig.* 4.) to deſcribe parabolic arcs as above; and let thoſe arcs be *HP*, *PQ*, *QR*, &c. And let the obliquity of the line of incidence *GH* to the firſt plane *Aa* be ſuch, that the ſine of incidence may be to the radius of the circle whoſe ſine it is, in the ſame ratio which the ſame ſine of incidence hath to the ſine of emergence from the plane *Dd* into the ſpace *DdeE*; and becauſe the ſine of emergence is now become equal to radius, the angle of emergence will be a right one, and therefore the line of emergence will coincide with the plane *Dd*. Let the body come to this plane in the point *R*; and becauſe the line of emergence coincides with that plane it is manifeſt that the body can proceed no farther towards the plane *Ee*. But neither can it proceed in the line of emergence *Rd*; becauſe it is perpetually attracted or impelled towards the medium of incidence. It will return therefore between the planes *Cc*, *Dd*, deſcribing an arc of a parabola *QRq*; whoſe principal vertex (by what *Galileo* has demonſtrated) is in *R*, cutting the plane *Cc* in the ſame angle at *q*, that it did before at *Q*; then going on in the parabolic arcs *qp*, *ph*, &c. ſimilar and equal to the former arcs *QP*, *PH*, &c. it will cut the reſt of the planes in the ſame angles at *p*, *h*, &c. as it did before in *P*, *H*, *&c*. and will emerge at laſt with the ſame obliquity at *h*, with which it firſt impinged on that plane at *H*. Conceive now the intervals of the planes *Aa*, *Bb*, *Cc*, *Dd*, *Ee*, &c. to be infinitely diminiſhed, and the number infinitely increaſed, ſo that the action of attraction or impulſe, exerted according to any aſſigned law, may become continual; and the angle of emergence remaining all along equal to the angle of incidence will be equal to the ſame alſo at laſt. *Q. E. D.*

SCHO-

SCHOLIUM.

Theſe attractions bear a great reſemblance to the reflexions and refractions of light, made in a given ratio of the ſecants, as was diſcovered by *Snellius*; and conſequently in a given ratio of the ſines, as was exhibited by *Des Cartes*. For it is now certain from the phænomena of *Jupiter*'s Satellits confirmed by the obſervations of different Aſtronomers, that light is propagated in ſucceſſion, and requires about ſeven or eight minutes to travel from the Sun to the Earth. Moreover the rays of light that are in our air (as lately was diſcovered by *Grimaldus*, by the admiſſion of light into a dark room through a ſmall hole, which I have alſo tried) in their paſſage near the angles of bodies whether tranſparent or opake (ſuch as the circular and rectangular edges of gold, ſilver and braſs coins, or of knives or broken pieces of ſtone or glaſs) are bent or inflected round thoſe bodies as if they were attracted to them; and thoſe rays which in their paſſage come neareſt to the bodies are the moſt inflected, as if they were moſt attracted; which thing I my ſelf have alſo carefully obſerved. And thoſe which paſs at greater diſtances are leſs inflected; and thoſe at ſtill greater diſtances are a little inflected the contrary way and form three fringes of colours. In *Pl.* 25. *Fig.* 6. *s* repreſents the edge of a knife or any kind of wedge *AsB*; and *gowog*, *fnunf*, *emtme*, *dlsld* are rays inflected towards the knife in the arcs *owo*, *nun*, *mtm*, *lsl*; which inflection is greater or leſs according to their diſtance from the knife. Now ſince this inflection of the rays is performed in the air without the knife, it follows that the rays which fall upon the knife are firſt inflected in the air before they touch the knife. And the caſe is the ſame of the rays falling upon glaſs. The refraction therefore is made, not in the point of incidence, but gradually, by a continual inflection of the rays; which is done partly in the air before they touch the glaſs, partly (if I miſtake not) within the glaſs, after they have entred it; as is repreſented (*Pl.* 25. *Fig.* 7.) in the rays $ckzc$, $biyb$, $ahxa$ falling upon r, q, p, and inflected between k and z, i and y, h and x. Therefore becauſe of the analogy there is between the propagation of the rays of light, and the motion of bodies, I thought it not amiſs to add the following propoſitions for optical uſes; not at all conſidering the nature of the rays of light, or enquiring whether they are bodies or not; but only determining the trajectories of bodies which are extremely like the trajectories of the rays.

PROPOSITION XCVII. PROBLEM XLVII.

Suppoſing the ſine of incidence upon any ſuperficies to be in a given ratio to the ſine of emergence; and that the inflection of the paths of thoſe bodies near that ſuperficies is performed in a very ſhort ſpace which may be conſidered as a point; it is required to determine ſuch a ſuperficies as may cauſe all the corpuſcles iſſuing from any one given place to converge to another given place.

Let A (*Pl.* 25. *Fig.* 8.) be the place from whence the corpuſcles diverge; B the place to which they ſhould converge; *CDE* the curve line which by its revolution round the axis *AB* deſcribes the ſuperficies ſought; *D*, *E*, any two points of that curve; and *EF*, *EG* perpendiculars let fall on the paths of the bodies *AD*, *DB*. Let the point *D* approach to and coaleſce with the point *E*; and the ultimate ratio of the line *DF* by which *AD* is increaſed, to the line *DG* by which *DB* is diminiſhed, will be the ſame as that of the ſine of incidence to the ſine of emergence. Therefore the ratio of the increment of the line *AD* to the decrement of the line *DB* is given; and therefore if in the axis *AB* there be taken any where the point *C* through which the curve *CDE* muſt paſs, and *CM* the increment of *AC* be taken in that given ratio to *CN* the decrement of *BC*, and from the centres *A*, *B*, with the intervals *AM*, *BN*, there be deſcribed two circles cutting each other in *D*; that point *D* will touch the curve ſought *CDE*, and by touching it any where at pleaſure, will determine that curve. *Q. E. I.*

Cor. 1. By cauſing the point *A* or *B* to go off ſometimes in infinitum, and ſometimes to move towards other parts of the point *C*, will be obtained all thoſe figures which *Carteſius* has exhibited in his Optics and Geometry relating to refractions. The invention of which *Carteſius* having thought fit to conceal, is here laid open in this propoſition.

Cor. 2. If a body lighting on any ſuperficies *CD* (*Pl.* 25. *Fig.* 9.) in the direction of a right line *AD*, drawn according to any law, ſhould emerge in the direction of another right line *DK*; and from the point *C* there be drawn curve lines *CP*, *CQ* always perpendicular to *AD*, *DK*; the increments of the lines *PD*, *QD*, and therefore the lines themſelves *PD*, *QD*, generated by thoſe increments, will be as the ſines of incidence and emergence to each other, and è contra.

Proposition XCVIII. Problem XLVIII.

The ſame things ſuppoſed; if round the axis AB *(*Pl. 25. Fig. 10.*) any attractive ſuperficies be deſcribed as* CD, *regular or irregular, through which the bodies iſſuing from the given place* A *muſt paſs; it is required to find a ſecond attractive ſuperficies* EF, *which may make thoſe bodies converge to a given place* B.

Let a line joining *AB* cut the firſt ſuperficies in *C* and the ſecond in *E*, the point *D* being taken any how at pleaſure. And ſuppoſing the ſine of incidence on the firſt ſuperficies to the ſine of emergence from the ſame, and the ſine of emergence from the ſecond ſuperficies to the ſine of incidence on the ſame, to be as any given quantity M to another given quantity N; then produce *AB* to *G*, ſo that *BG* may be to *CE* as M—N to N; and *AD* to *H*, ſo that *AH* may be equal to *AG*; and *DF* to *K* ſo that *DK* may be to *DH* as N to M. Join *KB*, and about the centre *D* with the interval *DH* deſcribe a circle meeting *KB* produced in *L*, and draw *BF* parallel to *DL*; and the point *F* will touch the line *EF*, which being turned round the axis *AB* will deſcribe the ſuperficies ſought. *Q. E. F.*

For conceive the lines *CP*, *CQ* to be every where perpendicular to *AD*, *DF*, and the lines *ER*, *ES* to FB, FD reſpectively, and therefore QS to be always equal to CE; and (by cor. 2. prop. 97.) PD will be to QD as M to N, and therefore as DL to DK, or FB to FK; and by diviſion as $DL-FB$ or $PH-PD-FB$ to FD or $FQ-QD$; and by compoſition as $PH-FB$ to FQ, that is, (becauſe PH and CG, QS and CE are equal) as $CE+BG-FR$ to $CE-FS$. But (becauſe BG is to CE, as M—N to N) it comes to paſs alſo that $CE+BG$ is to CE as M to N; and therefore, by diviſion, FR is to FS as M to N; and therefore (by cor. 2. prop. 97.) the ſuperficies EF compels a body, falling upon it in the direction DF, to go on in the line FR to the place B. *Q. E. D.*

SCHOLIUM.

In the ſame manner one may go on to three or more ſuperficies. But of all figures the ſphærical is the moſt proper for optical uſes. If the object glaſſes of teleſcopes were made of two glaſſes of a ſphærical figure, containing water between them; it is not unlikely that the errors of the refractions made in the extreme parts of the ſuperficies of the glaſſes, may be accurately enough corrected by the refractions of the water. Such object-glaſſes are to be preferred before elliptic and hyperbolic glaſſes, not only becauſe they may be formed with more eaſe and accuracy, but becauſe the pencils of rays ſituate without the axis of the glaſs would be more accurately refracted by them. But the different refrangibility of different rays is the real obſtacle that hinders optics from being made perfect by ſphærical or any other figures. Unleſs the errors thence ariſing can be corrected, all the labour ſpent in correcting the others is quite thrown away.

The End of the Firſt Volume.

MVSEVM
BRITAN
NICVM

Plate. XXV. Vol. I. P. 320.

Fig. 1. p. 311.

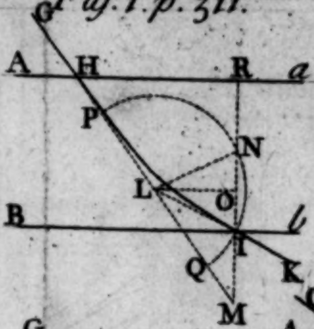


Fig. 2. p. 313.

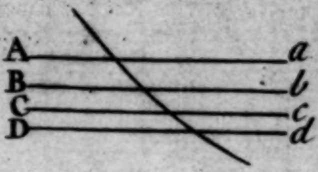


Fig. 3. p. 313.

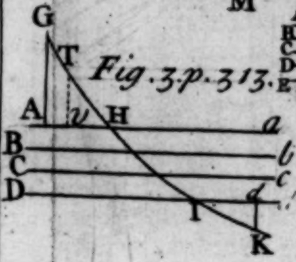


Fig. 4. p. 314.

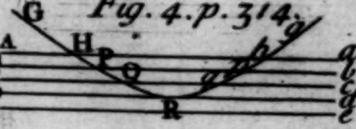


Fig. 5. p. 316.

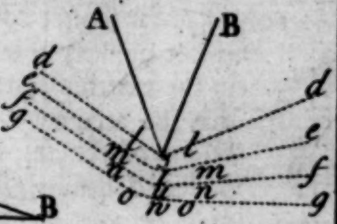


Fig. 7. p. 317.

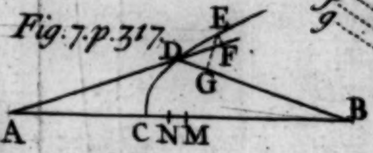


Fig. 6. p. 317.

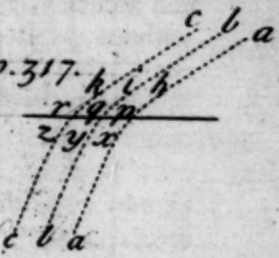


Fig. 8. p. 318.

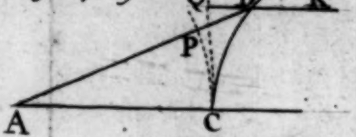


Fig. 9. p. 319.